

ZERO-DIVISOR GRAPH WITH SEVEN VERTICES

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ABSTRACT. Inspired by the work in [5] regarding the classification of all the zero-divisor graphs with six vertices, we obtain all the zero-divisor graphs with seven vertices. Hence we classify all the zero-divisor commutative semigroups with 8 elements. We also obtain all the connected graphs with seven vertices which satisfies the necessary condition \star of zero-divisor graphs given in [1] but are not the zero-divisor graphs.

1. INTRODUCTION

Let S be a commutative semigroup with zero. The *zero-divisor graph* of S , denoted $\Gamma(S)$, is the graph with vertices corresponding to the nonzero zero-divisors of S , and distinct zero-divisors x and y are adjacent if and only if $xy = 0$. A semigroup is called a *zero-divisor semigroup* if it consists solely of zero-divisors.

Given a connected graph G , let $V(G)$ denote the vertices set of G and $E(G)$ the edges set of G . For any two distinct vertices a and b , define $ab = 0$ if a and b are adjacent otherwise define $ab = ba \in V(G)$. Using this way, we can get a family of multiplication tables corresponding to G . If there exists a multiplication table which defines a semigroup, denoted $S(G)$, then G is a zero-divisor graph.

In [3], all the non-isomorphic connected graphs with six vertices are given. Based on [3], Sauer classified all the zero-divisor graphs with six vertices in [5].

The purpose of this paper is to extend Sauer's work [5] to zero-divisor graphs with seven vertices.

This paper is organized as follows. In section 2, we compare the necessary conditions for the zero-divisor graphs in [1] and conclude that condition (4) in Theorem 2.1 is the most important condition. In section 3, we give a method of filling the multiplication table of zero-divisor graphs and find some properties of zero-divisor graphs with seven vertices. In section 4, we get all the zero-divisor graphs with seven vertices in [4]. Hence we classify all the zero-divisor commutative semigroups with 8 elements. In section 5, we get all the non-zero divisor graphs in [4] which satisfy the necessary condition (4) in Theorem 2.1.

2. RELATIONS AMONG THE NECESSARY CONDITIONS REGARDING THE ZERO-DIVISOR GRAPHS GIVEN IN THEOREM 1 OF [1].

Definition 2.1. Given a connected graph G . Let a be a vertex of G . We define $N(a)$ be a set of all vertices which is adjacent to a and $\overline{N(a)} = N(a) \cup \{a\}$.

In [1, Theorem 1], the following necessary conditions for a zero-divisor graph were given.

Date: February 24, 2015.

Theorem 2.1. [1, Theorem 1] *If G is the graph of a semigroup then G satisfies all of the following conditions.*

- (1) G is connected.
- (2) Any two vertices of G are connected by a path with ≤ 3 edges.
- (3) If G contains a cycle then the core of G is a union of quadrilaterals and triangles, and any vertex not in the core of G is an end.
- (4) For any pair x, y of nonadjacent vertices of G , there is a vertex z with $N(x) \cup N(y) \subset \overline{N(z)}$.

In this section, we find the relations among those necessary conditions in Theorem 2.1.

Lemma 2.1. $N(a) \cup N(b) \subset \overline{N(c)}$ implies $d(a, b) \leq 3$.

Proof. If $N(a) \cup N(b) \subset \overline{N(c)} - N(c)$, then $c \in N(a)$ or $c \in N(b)$. Assume $c \in N(b)$ and $f \in N(a)$, then we have path $a - f - c - b$. Hence $d(a, b) \leq 3$.

If $N(a) \cup N(b) \subset N(c)$ and $e \in N(a) \cap N(b)$, then we have path $a - e - b$. Hence $d(a, b) \leq 3$.

Suppose $N(a) \cup N(b) \subset N(c)$ and $N(a) \cap N(b) = \emptyset$. If $d \in N(a)$ and $e \in N(b)$, then $\{a, c\} \subset N(d)$ and $\{b, c\} \subset N(e)$. Hence there exists a vertex f such that

$$\{a, b, c\} \subset N(d) \cup N(e) \subset \overline{N(f)}$$

Hence we have path $a - f - b$ and $d(a, b) \leq 3$. \square

Lemma 2.2. $d(a, b) \leq 3$ does not imply $N(a) \cup N(b) \subset \overline{N(c)}$.

Proof. The connected graph G with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{b, e, g\}$, $N(b) = \{a, c\}$, $N(c) = \{b, d\}$, $N(d) = \{c, f\}$, $N(e) = \{a, f\}$, $N(f) = \{d, e, g\}$, $N(g) = \{a, f\}$ is a counter example. \square

Lemma 2.3. $d(a, b) \leq 3$ does not imply a union of quadrilaterals and triangles.

Proof. The circuit $a - b - c - d - e - f - a$ gives a counter-example. \square

Lemma 2.4. a union of quadrilaterals and triangles does not imply $d(a, b) \leq 3$.

Proof.

$$\begin{aligned} N(a) &= \{b, c\}, N(b) = \{a, c, d, e\}, N(c) = \{a, b, f, g\}, N(d) = \{b, e, h\}, \\ N(e) &= \{b, d, f, i\}, N(f) = \{c, e, g, i\}, N(g) = \{c, f, j\}, N(h) = \{d, i, k\}, \\ N(i) &= \{e, f, h, j, k\}, N(j) = \{g, i, k\}, N(k) = \{h, i, j\} \end{aligned}$$

gives a counter-example. \square

Lemma 2.5. $N(a) \cup N(b) \subset \overline{N(c)}$ implies a union of quadrilaterals and triangles.

Proof. Suppose the induced subgraph is a circuit $a_1 a_2 a_3 \cdots a_k a_1$. Then $\{a_2, a_k\} \subset N(a_1)$, and $\{a_2, a_4\} \subset N(a_3)$. It follows that there exists a vertex f such that $N(a_1) \cup N(a_3) \subset \overline{N(f)}$. Since the induced subgraph is a circuit $a_1 a_2 a_3 \cdots a_k a_1$, we get $f \neq a_1$, $f \neq a_2$, $f \neq a_3$. Hence $a_1 a_2$ is an edge of quadrilateral $a_1 a_2 f a_k a_1$ and $a_2 a_3$ is an edge of quadrilateral $a_2 a_3 f a_k a_2$. We have a short circuit $a_k f a_4 \cdots a_{k-1} a_k$. Continue this process we get $a_1 a_2 a_3 \cdots a_k a_1$ is a union of quadrilaterals and triangles. \square

Hence condition (4) in Theorem 2.1 is the most important necessary condition. Henceforth, we call condition (4) in Theorem 2.1 the condition \star .

3. A WAY TO FILL THE MULTIPLICATION TABLE OF ZERO-DIVISOR GRAPHS AND SOME PROPERTIES OF ZERO-DIVISOR GRAPHS WITH SEVEN VERTICES

3.1. a way to fill the multiplication table of zero-divisor graphs.

Remark 3.1. For any $\{a, b\} \in V(G)$, if $a \neq b$, then $ab = 0$ if and only if $a - b$ is an edge of G and $ab \in V(G)$ if and only if $a - b$ is not an edge of G . For any $a \in V(G)$, $a^2 \in V(G) \cup \{0\}$.

Definition 3.1. Let G be a connected graph G with $V(G) = \{a, b, c, d, e, f, g\}$. Let $x \in V(G)$. The spectrum of x is defined to be $\text{Spec}(x) = (xa, xb, xc, xd, xe, xf, xg)$. Given $y, z \in V(G)$, define $yz = x$ if $\text{Spec}(yz) = \text{Spec}(x)$.

Definition 3.2. Let a and b be non-adjacent vertices of G . Define $D(ab)$ to be a set of vertices such that $c \in D(ab)$ if and only if $N(a) \cup N(b) \subset \overline{N(c)}$. For vertex $a \in V(G)$, define $D(a^2)$ to be a set such that $c \in D(a^2)$ if and only if $c = 0$ or $N(a) \subset \overline{N(c)}$.

Example 3.1. The graph G defined by $N(a) = N(b) = \{x, y\}$, $N(c) = \{y\}$, $N(x) = \{a, b, y, z, w\}$, $N(y) = \{a, b, c, x\}$, $N(z) = \{x\}$, $N(w) = \{x\}$ is a zero-divisor graph because by Light's associativity test, one can show the multiplication given by following table is associative.

•	a	b	c	x	y	z	w
a	a	a	a	0	0	a	a
b	a	b	a	0	0	a	a
c	a	a	c	x	0	a	a
x	0	0	x	x	0	0	0
y	0	0	0	0	y	y	y
z	a	a	a	0	y	z	z
w	a	a	a	0	y	z	w

$\text{Spec}(a) = (a, a, a, 0, 0, a, a)$, $\text{Spec}(b) = (a, b, a, 0, 0, a, a)$, $\text{Spec}(c) = (a, a, c, x, 0, a, a)$, $\text{Spec}(x) = (0, 0, x, x, 0, 0, 0)$, $\text{Spec}(y) = (0, 0, 0, 0, y, y, y)$, $\text{Spec}(z) = (a, a, a, 0, y, z, z)$, $\text{Spec}(w) = (a, a, a, 0, y, z, w)$.

$D(ab) = \{a, b, y\}$ because

$$N(a) \cup N(b) = \{x, y\} \subset N(a) \cap N(b) \cap \overline{N(y)}$$

We define $ab = a$ because

- (1) if $ab = b$, then $(ba)c = bc = a \neq b = ba = b(ac)$;
- (2) if $ab = y$, then $y^2 = 0$. It follows that $a^2b = ab = y \neq 0 = ay = a(ab)$.

3.2. some properties of zero-divisor graphs with seven vertices.

Lemma 3.1. Let G be a zero-divisor graph and $d(x) = \Delta$, where $d(x) = |N(x)|$ and Δ is the maximum degree. Then for any vertex $y \in V(G)$, $d(x, y) \leq 2$.

Proof. By [2, Lemma 4.1], we get $N(y) \subseteq N(x)$ for all $y \notin N(x)$. □

Lemma 3.2. Let G be a zero-divisor graph with seven vertices. If $d(x) = \Delta$, then $d(x) \geq 4$.

Proof. Suppose $\Delta = 3$. Let $N(a) = \{x, y, z\}$. Since $|N(a)| = 3$, one gets that $N(b)$, $N(c)$ and $N(w)$ are subsets of $N(a)$. Suppose $N(x) = \{a, b, c\}$. Since $|N(x)| = 3$, one gets $w \in N(y)$ or $w \in N(z)$ and $x \notin N(y)$ or $x \notin N(z)$. It follows that $N(y)$ is not a subset of $N(x)$ or $N(z)$ is not a subset of $N(x)$. This is a contradiction. Hence one gets $N(x) \cap \{a, b, c\} = N(y) \cap \{a, b, c\} = N(z) \cap \{a, b, c\} = 2$. If $d(x) = 3$, then one get xy or xz is an edge. Hence $d(y) = 3$ or $d(z) = 3$. If xy is an edge, then xz is not an edge. But $N(z)$ is not a subset of $N(x)$. Contradiction. Hence $|N(x)| = |N(y)| = |N(z)| = 2$. Let $N(b) = \{z\}$, $N(c) = \{y\}$, $N(w) = \{x\}$. Then there is no v such that $N(y) \cup N(z) \subseteq \overline{N(v)}$. The proof is completed. \square

Lemma 3.3. *Let G be a connected graph satisfies the necessary condition \star . Let $d(x) = \Delta$. Then emanating from x an end w results a graph satisfies the necessary condition \star .*

Proof. If x and y are not adjacent, then $N(y) \subseteq N(x)$ by [2]. Hence $N(y) \cup N(w) \subset N(x)$. If $y \in N(x)$, then $N(y) \cup N(w) = N(y) \cup \{x\} = N(y)$. \square

4. ZERO DIVISOR GRAPH WITH SEVEN VERTICES

4.1. zero-divisor graphs with seven vertices produced by applying Theorems in [1] and [5].

Theorem 4.1. [1, Theorem 3] *The following graphs are the graph of a semigroup.*

- (1) *A complete graph or a complete graph together with one end.*
- (2) *A complete bipartite graph or a complete bipartite graph together with an end.*
- (3) *A refinement of a star graph.*
- (4) *A graph which is the union of two star graphs whose centers are connected by a single edge.*

Theorem 4.2. (1) [5, Theorem IV.1] *A complete graph together with any number of ends, each of which emanates from one of two vertices, is the graph of a commutative semigroup.*

- (2) [5, Theorem IV.2] *The complete graph on three vertices, from each of which emanates at least one end, is the graph of a commutative semigroup.*
- (3) [5, Theorem IV.3] *The complete graph on four or more vertices together with any number of ends emanating from at least three different vertices is never the graph of a commutative semigroup.*
- (4) [5, Theorem IV.4] *A complete bipartite graph together with any number of ends emanating from the same vertex is the graph of a commutative semigroup.*
- (5) [5, Theorem IV.5] *A complete bipartite graph together with two or more ends emanating from at least two distinct vertices is never the graph of a commutative semigroup.*

Theorem 4.3. [5] *Up to isomorphism, there are 67 zero-divisor graphs with six vertices. They are one star graph, 33 refinements of a star graph, two double star graphs, two complete bipartite graphs, four complete bipartite graphs with ends emanating from at most two different vertices, a complete graphs with three vertices and three ends emanating from three different vertices, and twenty-four exceptional cases.*

4.2. a seven vertices zero-divisor graph produced by applying [2, Lemma 3.14].

Theorem 4.4. [2, Lemma 3.14] Suppose G is a zero divisor graph of a semigroup, $x \in V(G)$, $y \notin V(G)$. Let G' be a graph defined by $V(G') = V(G) \cup \{y\}$, $N(y) = N(x)$ if $x^2 \neq 0$, or $N(y) = \overline{N(x)}$ if $x^2 = 0$. Then G' is a zero-divisor graph of a semigroup.

Proof. Define $xy = x^2 = y^2$ and $xz = yz$ for all $z \in V(G) - \{y\}$. \square

4.3. list of all the zero-divisor graphs with seven vertices.

Theorem 4.5. The following graphs are all the zero-divisor graphs in [4]. Notice there exists a one-to-one correspondence between zero-divisor graphs and zero-divisor semigroups. Hence we give a classification of all the zero-divisor commutative semigroups with 8 elements.

$G270 - G272, G314 - G317, G319, G379 - G382, G384, G388, G390, G392 -$
 $G393, G411, G473 - G474, G476 - G481, G483, G485 - G486, G493, G503, G507, G513,$
 $G522, G525, G551, G598 - G599, G601, G603 - G604, G606,$
 $G612 - G614, G616, G618 - G620, G624, G626, G629,$
 $G631, G633, G639, G667 - G668, G670, G671 - G672, G678, G740 - G741, G743, G746 -$
 $G749, G751 - G753, G755, G757 - G759, G762, G764, G767, G775,$
 $G780, G786, G790 - G792, G794 - G796, G798, G800 - G801, G805, G812 - G815, G832,$
 $G872, G884 - G891, G894, G896 - G898, G902, G906, G908 - G909, G913 - G916,$
 $G919 - G925, G927, G929 - G930, G932, G934, G939, G944, G948, G950 - G952, G957,$
 $G972, G975, G1007 - G1009, G1012 - G1020, G1025 - G1029, G1031 - G1032,$
 $G1035 - G1042, G1045 - G1050, G1052 - G1053, G1056 - G1057, G1059,$
 $G1062, G1067, G1072, G1077 - G1081, G1085, G1088, G1106, G1108 - G1111,$
 $G1113 - G1119, G1121 - G1126, G1128 - G1129, G1131 - G1132, G1134 - G1135,$
 $G1137 - G1145, G1147 - G1152, G1157, G1163, G1169, G1173 - G1176,$
 $G1178 - G1200, G1202 - G1203, G1205 - G1208, G1210, G1213 - G1252,$

Proof. The proof is given in the following examples in this section. \square

Example 4.1. $G270 - G272$, are zero-divisor graphs. ($G270$ is a star graph. $G271$ and $G272$ are bi-star graphs).

Example 4.2. $G314 - G317$ are zero-divisor graphs (complete graphs with three vertices emanating ends).

Example 4.3. Let $G319$ be a graph with $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$, and $E(G)$ is defined by $N(1) = \{2, 3\}$, $N(2) = N(3) = \{1, 4\}$, $N(4) = \{2, 3, 5, 6, 7\}$, $N(5) = N(6) = N(7) = \{4\}$. Then G is a zero-divisor graph since G is a complete bipartite graph together with ends emanating from the same vertex.

	●	1	2	3	4	5	6	7
	1	1	0	0	4	4	4	4
	2	0	2	2	0	2	2	2
	3	0	2	2	0	2	2	2
G319	4	4	0	0	0	0	0	0
	5	4	2	2	0	2	2	2
	6	4	2	2	0	2	2	2
	7	4	2	2	0	2	2	2

Example 4.4. $G379$ is a zero-divisor graph because it is the refinement of a star graph with seven vertices..

Example 4.5. Let $G380$ be a connected graph with six vertices $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and the edges set $E(G)$ is defined by the following way, $N(1) = \{2\}$, $N(2) = \{1, 3, 6, 7\}$, $N(3) = \{2, 4, 5, 6, 7\}$, $N(4) = N(5) = \{3\}$, $N(6) = N(7) = \{2, 3\}$. $G380$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	7	7	7	7
	2	0	2	0	2	2	0	0
G380	3	3	0	3	0	0	0	0
	4	7	2	0	4	4	7	7
	5	7	2	0	4	4	7	7
	6	7	0	0	7	7	7	7
	7	7	0	0	7	7	7	7

Example 4.6. $G381$ is a zero-divisor graph.

	•	a	b	c	x	y	z	w
	a	a	a	a	0	0	0	a
	b	a	a	a	0	0	x	a
G381	c	a	a	c	0	y	y	c
	x	0	0	0	0	0	x	0
	y	0	0	y	0	y	y	y
	z	0	x	y	x	y	z	y
	w	0	0	y	0	y	y	w

Example 4.7. $G382$ is a zero-divisor graph.

	•	a	b	c	x	y	z	w
	a	0	z	z	0	0	0	z
	b	z	b	b	0	z	z	b
G382	c	z	b	b	0	z	z	b
	x	0	0	0	x	x	0	0
	y	0	z	z	x	x	0	z
	z	0	z	z	0	0	0	z
	w	z	b	b	0	z	z	b

Example 4.8. $G384$ is a zero-divisor graph by applying [2, Lemma 3.14] to $G113$ since $5^2 \neq 0$.

	•	1	2	3	4	5	6	7
	1	4	0	0	4	2	2	4
	2	0	0	0	0	2	2	0
	3	0	0	3	0	3	3	3
G384	4	4	0	0	4	0	0	4
	5	2	2	3	0	5	5	3
	6	2	2	3	0	5	5	3
	7	4	0	3	4	3	3	7

Example 4.9. $G388$ is a refinement of a star graph and hence a zero divisor graph.

Example 4.10. $G390$ is a zero-divisor graph.

	•	a	b	c	x	y	z	w
	a	0	a	a	0	0	0	a
	b	a	b	b	0	0	a	b
	c	a	b	b	0	x	a	b
G390	x	0	0	0	0	x	0	0
	y	0	0	x	x	y	x	x
	z	0	a	a	0	x	0	a
	w	a	b	b	0	x	a	b

Example 4.11. The connected graph $G392$ with $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and $E(G)$ defined by $N(1) = N(2) = \{3\}$, $N(3) = \{1, 2, 4, 5, 6\}$, $N(4) = N(5) = N(6) = \{3, 7\}$, $N(7) = \{4, 5, 6\}$. $G392$ is a zero-divisor graph by [5, Theorem IV.4].

	•	1	2	3	4	5	6	7
	1	4	4	0	4	4	4	3
	2	4	4	0	4	4	4	3
	3	0	0	0	0	0	0	3
G392	4	4	4	0	4	4	4	0
	5	4	4	0	4	4	4	0
	6	4	4	0	4	4	4	0
	7	3	3	3	0	0	0	7

Example 4.12. $G393$ is a zero-divisor graph. $G393$ can be defined by the following way: $N(1) = \{2\}$, $N(2) = \{1, 3, 5, 7\}$, $N(3) = \{2, 4\}$, $N(4) = \{3, 5\}$, $N(5) = \{2, 4, 6, 7\}$, $N(6) = \{5\}$, $N(7) = \{2, 5\}$.

We define the multiplication table in the following way. One can check the associativity by Light's test.

	•	1	2	3	4	5	6	7
	1	3	0	3	2	5	7	5
	2	0	0	0	2	0	2	0
	3	3	0	3	0	5	5	5
G393	4	2	2	0	4	0	4	2
	5	5	0	5	0	0	0	0
	6	7	2	5	4	0	4	2
	7	5	0	5	2	0	2	0

Example 4.13. Let $G411$ be a graph with $V(G) = \{a, b, c, x, y, z, w\}$, and $E(G)$ is defined by $N(a) = \{x, y, z, w\}$, $N(b) = N(c) = \{x, y\}$, $N(x) = N(y) = \{a, b, c\}$, $N(z) = N(w) = \{a\}$. Then G is a zero-divisor graph since G is a complete bipartite graph together with ends emanating from the same vertex. $G411$ is a zero-divisor graph by [5, Theorem IV.4] (bi-partite graph).

Example 4.14. The connected graph $G473$ which is defined by $N(a) = \{b, c, x, y, z, w\}$, $N(b) = \{a, c, x\}$, $N(c) = \{a, b, x\}$, $N(x) = \{a, b, c\}$, $N(y) = N(z) = N(w) = \{a\}$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.15. Let $G474$ be a graph with $V(G) = \{a, b, c, x, y, z, w\}$, and $E(G)$ is defined by $N(a) = \{c, x, y, z\}$, $N(b) = N(w) = \{x\}$, $N(c) = \{a, x, y\}$, $N(x) = \{a, b, c, w, y\}$, $N(y) = \{a, c, x\}$, $N(z) = \{a\}$. Then G is a zero-divisor graph since G is a complete graph together with ends emanating from one of the vertices.

Example 4.16. $G476$ is a refinement of a star graph and hence a zero divisor graph.

Example 4.17. $G477$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	4	3	3	3
	2	0	2	0	0	2	0	0
G477	3	3	0	3	0	3	3	3
	4	4	0	0	4	0	0	0
	5	3	2	3	0	5	3	3
	6	3	0	3	0	3	3	3
	7	3	0	3	0	3	3	3

Example 4.18. $G478$ is a refinement of a star graph and hence a zero divisor graph.

Example 4.19. $G479$ is a zero-divisor graph.

	•	a	b	c	x	y	z	w
	a	0	a	a	0	0	0	a
	b	a	b	b	0	0	a	b
G479	c	a	b	c	x	0	a	b
	x	0	0	x	x	0	0	0
	y	0	0	0	0	y	y	y
	z	0	a	a	0	y	y	z
	w	a	b	b	0	y	z	w

Example 4.20. Let $G480$ be a connected graph with six vertices $V(G) = \{a, b, c, x, y, z, w\}$ and the edges set $E(G)$ is defined by the following way, $N(a) = \{x, y, z\}$, $N(b) = N(c) = \{x, y\}$, $N(x) = \{a, b, c, w, y\}$, $N(y) = \{a, b, c, x\}$, $N(z) = \{a\}$, $N(w) = \{x\}$. $G480$ is a zero-divisor graph (combining $G113$ and [2, Lemma 3.14] since $b^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	a	a	a	0	0	0	a
	b	a	a	a	0	0	x	a
G480	c	a	a	a	0	0	x	a
	x	0	0	0	0	0	x	0
	y	0	0	0	0	y	y	y
	z	0	x	x	x	y	z	y
	w	a	a	a	0	y	y	w

Example 4.21. Let $G481$ be a connected graph with six vertices $V(G) = \{a, b, c, x, y, z\}$ and the edges set $E(G)$ is defined by the following way, $N(a) = \{x, y, z\}$, $N(b) = \{x, y\}$, $N(c) = \{y\}$, $N(x) = \{a, b, y, z\}$, $N(y) = \{a, b, c, w, x\}$, $N(z) = \{a, x\}$.

$G481$ is a zero-divisor graph $G481$ by checking the following multiplication table.

G481

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	a
b	a	b	b	0	0	a	b
c	a	b	c	x	0	a	c
x	0	0	x	x	0	0	x
y	0	0	0	0	y	y	0
z	0	a	a	0	y	y	a
w	a	b	c	x	0	a	c

Example 4.22. $G483$ is a zero-divisor graph (combining $G119$ and [2, Lemma 3.14] because $y^2 \neq 0$).

G483

•	a	b	c	x	y	z	w
a	0	z	z	0	0	0	0
b	z	b	b	0	z	z	z
c	z	b	b	0	z	z	z
x	0	0	0	x	x	0	x
y	0	z	z	x	x	0	x
z	0	z	z	0	0	0	0
w	0	z	z	x	x	0	x

Example 4.23. $G485$ is a zero-divisor graph.

G485

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	b	b	0	0	z	b
c	a	b	b	0	x	z	b
x	0	0	0	0	x	0	0
y	0	0	x	x	y	0	x
z	0	z	z	0	0	z	z
w	a	b	b	0	x	z	b

Example 4.24. $G486$ is a zero-divisor graph.

G486

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	b	a	0	0	0	b
c	a	a	c	0	z	z	c
x	0	0	0	x	x	0	0
y	0	0	z	x	y	z	z
z	0	0	z	0	z	z	z
w	a	b	c	0	z	z	w

Example 4.25. $G493$ is a zero-divisor graph.

G493

•	1	2	3	4	5	6	7
1	1	0	3	5	5	5	3
2	0	2	0	2	0	0	2
3	3	0	3	0	0	0	3
4	5	2	0	4	5	5	2
5	5	0	0	5	5	5	0
6	5	0	0	5	5	5	0
7	3	2	3	2	0	0	7

Example 4.26. $G503$ is a refinement of star graph hence it is a zero divisor graph.

Example 4.27. The connected graph $G507$ with $V(G) = \{a_1, a_2, b_1, b_2, b_3, b_4, x_1\}$ and $E(G)$ defined by $N(a_1) = \{b_1, b_2, b_3, b_4, x_1\}$, $N(a_2) = \{b_1, b_2, b_3, b_4\}$, $N(b_1) = N(b_2) = N(b_3) = N(b_4) = \{a_1, a_2\}$, $N(x_1) = \{a_1\}$ is a zero-divisor graph by [5, Theorem IV.4].

Example 4.28. $G513$ is a zero-divisor graph.

G513

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	a
b	a	b	b	0	0	a	b
c	a	b	c	0	0	a	b
x	0	0	0	0	x	0	0
y	0	0	0	x	y	x	x
z	0	a	a	0	x	0	a
w	a	b	b	0	x	a	b

Example 4.29. $G522$ is a zero-divisor graph.

G522

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	0
b	a	b	b	0	0	a	a
c	a	b	c	0	0	a	a
x	0	0	0	0	x	0	x
y	0	0	0	x	y	x	y
z	0	a	a	0	x	0	x
w	0	a	a	x	y	x	y

Another way to prove that $G522$ is a zero-divisor graph is by combining $G118$ and [2, Lemma 3.14].

Example 4.30. The connected graph $G525$ with $V(G) = \{a_1, a_2, b_1, b_2, b_3, b_4, x_1\}$ and $E(G)$ defined by $N(a_1) = N(a_2) = \{b_1, b_2, b_3, b_4\}$, $N(b_1) = \{a_1, a_2, x_1\}$, $N(b_2) = N(b_3) = N(b_4) = \{a_1, a_2\}$, $N(x_1) = \{b_1\}$ is a zero-divisor graph by [5, Theorem IV.4].

Example 4.31. $G551$ is a refinement of a star graph and hence a zero-divisor graph.

Example 4.32. $G598$ is refinement of a star graph and hence a zero-divisor graph.

Example 4.33. $G599$ is a zero-divisor graph.

G599

•	1	2	3	4	5	6	7
1	1	0	3	4	3	6	6
2	0	2	0	0	2	0	0
3	3	0	3	0	3	6	6
4	4	0	0	4	0	0	0
5	3	2	3	0	5	6	6
6	6	0	6	0	6	0	0
7	6	0	6	0	6	0	0

Example 4.34. $G601$ is a zero-divisor graph.

G601

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	b	a	x	0	0	a
c	a	a	c	0	y	z	c
x	0	x	0	x	0	0	0
y	0	0	y	0	y	0	y
z	0	0	z	0	0	z	z
w	a	a	c	0	y	z	c

Example 4.35. $G603$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.36. Let $G604$ be a connected graph with seven vertices $V(G) = \{1, 2, 3, 4, 5, 6, 7\}$ and the edges set $E(G)$ is defined by the following way, $N(1) = \{2\}$, $N(2) = \{1, 3, 5, 6\}$, $N(3) = N(5) = \{2, 4, 6\}$, $N(4) = \{3, 5, 6\}$, $N(6) = \{2, 3, 4, 5, 7\}$, and $N(7) = \{6\}$. $G604$ is a zero-divisor graph.

G604

•	1	2	3	4	5	6	7
1	1	0	5	6	5	6	5
2	0	2	0	2	0	0	2
3	5	0	5	0	5	0	5
4	6	2	0	2	0	0	2
5	5	0	5	0	5	0	5
6	6	0	0	0	0	0	0
7	5	2	5	2	5	0	7

Example 4.37. $G606$ is a zero-divisor graph (combining $G159$ and [2, Lemma 3.14]).

G606

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	b	a	0	0	0	a
c	a	a	c	0	z	z	c
x	0	0	0	x	x	0	0
y	0	0	z	x	x	0	z
z	0	0	z	0	0	0	z
w	a	a	c	0	z	z	c

Example 4.38. $G612 - G613$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.39. G_{614} is a zero-divisor graph (combining G_{137} and [2, Lemma 3.14] because $b^2 \neq 0$.)

G_{614}

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	a
b	a	b	b	0	0	a	b
c	a	b	c	x	0	a	b
x	0	0	x	x	0	0	0
y	0	0	0	0	y	y	0
z	0	a	a	0	y	y	a
w	a	b	b	0	0	a	b

Example 4.40. G_{616} is a zero-divisor graph (combining G_{137} and [2, Lemma 3.14] since $z \times z \neq 0$.)

G_{616}

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	0
b	a	b	b	0	0	a	a
c	a	b	c	x	0	a	a
x	0	0	x	x	0	0	0
y	0	0	0	0	y	y	y
z	0	a	a	0	y	y	y
w	0	a	a	0	y	y	y

Example 4.41. $G_{618} - G_{619}$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.42. G_{620} is a zero-divisor graph (combining G_{166} and [2, Lemma 3.14].)

G_{620}

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	a	a	0	x	0	a
c	a	a	c	0	0	z	c
x	0	0	0	0	x	0	0
y	0	x	0	x	y	0	x
z	0	0	z	0	0	z	z
w	a	a	c	0	x	z	c

Example 4.43. Let G_{624} be a connected graph with six vertices $V(G) = \{a, b, c, x, y, z, w\}$ and the edges set $E(G)$ is defined by the following way, $N(a) = \{x, y, z\}$, $N(b) = \{x, z\}$, $N(c) = \{x, y\}$, $N(x) = \{a, b, c, z\}$, $N(y) = \{a, c, z\}$, $N(z) = \{a, b, x, y, w\}$, and $N(w) = \{z\}$. G_{624} is a zero-divisor graph.

G624

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	a	a	0	x	0	b
c	a	a	c	0	0	z	a
x	0	0	0	0	x	0	x
y	0	x	0	x	y	0	y
z	0	0	z	0	0	z	0
w	a	b	a	x	y	0	w

Example 4.44. Let $G626$ be a connected graph with six vertices $V(G) = \{a, b, c, x, y, z\}$ and the edges set $E(G)$ is defined by the following way, $N(a) = \{c, x, y\}$, $N(b) = \{y, z\}$, $N(c) = \{a, x, y\}$, $N(x) = \{a, c, y, z, w\}$, $N(y) = \{a, b, c, x\}$, $N(z) = \{b, x\}$, and $N(w) = \{x\}$.

$G626$ is a zero-divisor graph.

G626

•	a	b	c	x	y	z	w
a	0	x	0	0	0	y	y
b	x	b	x	x	0	0	x
c	0	x	0	0	0	y	y
x	0	x	0	0	0	0	0
y	0	0	0	0	0	y	y
z	y	0	y	0	y	z	z
w	y	x	y	0	y	z	z

Example 4.45. $G629$ is a zero-divisor graph.

G629

•	a	b	c	x	y	z	w
a	b	b	b	0	0	0	b
b	b	b	b	0	0	0	b
c	b	b	b	0	0	x	b
x	0	0	0	0	0	x	0
y	0	0	0	0	y	y	y
z	0	0	x	x	y	z	y
w	b	b	b	0	y	y	w

Example 4.46. $G631$ is a zero-divisor graph.

G631

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	a
b	a	b	b	0	0	a	b
c	a	b	b	0	0	a	b
x	0	0	0	0	x	0	0
y	0	0	0	x	y	0	x
z	0	a	a	0	0	0	a
w	a	b	b	0	x	a	b

Example 4.47. $G633$ is a zero-divisor graph.

G633

•	1	2	3	4	5	6	7
1	1	0	3	7	5	3	7
2	0	2	0	2	0	2	0
3	3	0	0	0	3	0	0
4	7	2	0	4	0	2	7
5	5	0	3	0	5	3	0
6	3	2	0	2	3	2	0
7	7	0	0	7	0	0	7

Example 4.48. $G639$ is a zero-divisor graph by [5, IV.1] and [2, Lemma 3.14].

G639

•	1	2	3	4	5	6	7
1	1	0	0	1	1	1	0
2	0	2	0	0	2	0	2
3	0	0	3	3	0	3	0
4	1	0	3	4	1	4	0
5	1	2	0	1	5	1	2
6	1	0	3	4	1	4	0
7	0	2	0	0	2	0	2

Example 4.49. The connected graph $G667$ with $V(G) = \{a_1, a_2, b_1, b_2, b_3, b_4, x_1\}$ and $E(G)$ defined by $N(a_1) = N(a_2) = N(a_3) = \{b_1, b_2, b_3\}$, $N(b_1) = \{a_1, a_2, a_3, x_1\}$, $N(b_2) = N(b_3) = \{a_1, a_2, a_3\}$, $N(x_1) = \{b_1\}$ is a zero-divisor graph by [5, IV.4].

Example 4.50. $G668$ is a zero-divisor graph because it is a refinement of star graph.

Example 4.51. The connected graph $G670$ which is defined by $N(a) = N(b) = \{c, d, e, f, g\}$, $N(c) = N(d) = N(e) = N(f) = N(g) = \{a, b\}$ is a zero-divisor graph by [5, IV.4].

Example 4.52. $G671 - G672$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.53. $G678$ is a zero-divisor graph by applying [2, Lemma 3.14] to $G119$ because $b^2 \neq 0$.

G678

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	a
b	a	b	b	0	0	a	b
c	a	b	c	0	0	a	b
x	0	0	0	0	x	0	0
y	0	0	0	x	y	x	0
z	0	a	a	0	x	0	a
w	a	b	b	0	0	a	b

Example 4.54. $G740$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.55. $G741$ is a zero-divisor graph. The proof of [5, Theorem IV.1] is wrong.

G741

•	1	2	3	4	5	6	7
1	1	0	1	0	0	1	1
2	0	0	0	0	0	5	4
3	1	0	1	0	0	1	1
4	0	0	0	0	0	0	4
5	0	0	0	0	0	5	0
6	1	5	1	0	5	6	1
7	1	4	1	4	0	1	7

Example 4.56. $G743$ is a zero-divisor graph.

G743

•	a	b	c	x	y	z	w
a	0	a	0	0	0	0	a
b	a	b	a	0	a	a	b
c	0	a	0	0	0	0	a
x	0	0	0	x	x	0	0
y	0	a	0	x	x	0	a
z	0	a	0	0	0	0	a
w	a	b	a	0	a	a	b

Example 4.57. $G746 - G747$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.58. $G748$ is a zero-divisor graph.

G748

•	1	2	3	4	5	6	7
1	1	0	5	4	5	0	5
2	0	0	0	0	0	0	2
3	5	0	5	0	5	2	3
4	4	0	0	0	0	0	0
5	5	0	5	0	5	0	5
6	0	0	2	0	0	2	6
7	5	2	3	0	5	6	7

Example 4.59. $G749$ is a zero-divisor graph (applying [2, Lemma 3.14] to $G157$ since $b^2 \neq 0$).

G749

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	b	a	x	0	0	b
c	a	a	c	0	y	z	a
x	0	x	0	x	0	0	x
y	0	0	y	0	y	0	0
z	0	0	z	0	0	z	0
w	a	b	a	x	0	0	a

Example 4.60. $G751$ is a zero-divisor graph.

G751

•	1	2	3	4	5	6	7
1	1	0	0	0	5	0	1
2	0	2	0	4	6	6	4
3	5	0	5	0	5	2	3
4	4	0	0	0	0	0	0
5	5	0	5	0	5	0	5
6	0	0	2	0	0	2	6
7	5	2	3	0	5	6	7

Example 4.61. $G752$ and $G753$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.62. $G755$ is a zero-divisor graph.

G755

•	1	2	3	4	5	6	7
1	1	0	3	4	6	6	7
2	0	2	0	0	2	0	0
3	3	0	4	0	4	4	0
4	4	0	0	0	0	0	0
5	6	2	4	0	2	0	4
6	6	0	4	0	0	0	4
7	7	0	0	0	4	4	4

Example 4.63. $G757$ is a zero-divisor graph.

G757

•	1	2	3	4	5	6	7
1	1	0	3	5	5	6	7
2	0	2	0	2	0	0	0
3	3	0	6	0	0	6	5
4	5	2	0	2	0	0	5
5	5	0	0	0	0	0	5
6	6	0	6	0	0	6	0
7	7	0	5	5	5	0	7

Example 4.64. $G758$ is a zero-divisor graph.

G758

•	1	2	3	4	5	6	7
1	1	0	3	7	5	5	7
2	0	2	0	2	0	2	0
3	3	0	3	0	5	5	0
4	7	2	0	2	0	2	0
5	5	0	5	0	0	0	0
6	5	2	5	2	0	2	0
7	7	0	0	0	0	0	0

Example 4.65. $G759$ is a zero-divisor graph (proved in Example 4.32).

G759

•	a	b	c	x	y	z	w
a	0	a	0	0	0	0	0
b	a	b	c	0	0	a	c
c	0	c	c	0	0	0	c
x	0	0	0	0	x	0	x
y	0	0	0	x	y	x	y
z	0	a	0	0	x	0	x
w	0	c	c	x	y	x	w

Example 4.66. $G762$ is a zero-divisor graph by applying [2, Lemma 3.14] to $G141$ since $b^2 \neq 0$.

G762

•	a	b	c	x	y	z	w
a	b	b	b	0	0	0	b
b	b	b	b	0	0	0	b
c	b	b	c	x	0	0	c
x	0	0	x	x	0	0	x
y	0	0	0	0	0	y	0
z	0	0	0	0	y	z	y
w	b	b	c	x	0	y	c

Example 4.67. $G764$ is a zero-divisor graph by applying [2, Lemma 3.14] to $G118$ since $a^2 = 0$.

G764

•	a	b	c	x	y	z	w
a	0	a	0	0	0	0	a
b	a	b	a	0	0	a	b
c	0	a	0	0	0	0	a
x	0	0	0	0	x	0	0
y	0	0	0	x	y	x	x
z	0	a	0	0	x	0	a
w	a	b	a	0	x	a	b

Example 4.68. $G767$ is a zero-divisor graph.

G767

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	a	a	0	0	0	a
c	a	a	c	x	0	0	a
x	0	0	x	x	0	0	0
y	0	0	0	0	y	y	y
z	0	0	0	0	y	y	y
w	a	a	a	0	y	y	w

Example 4.69. $G775$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.70. $G780$ is a zero-divisor graph (combining $G140$ and [2, Lemma 3.14] since $a^2 \neq 0$).

G780

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	a	a	0	0	0	a
c	a	a	a	0	0	0	a
x	0	0	0	0	x	0	0
y	0	0	0	x	y	z	z
z	0	0	0	0	z	z	z
w	a	a	a	0	z	z	w

Example 4.71. $G786$ is a zero-divisor graph (combining $G141$ and [2, Lemma 3.14] since $y^2 \neq 0$).

G786

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	b	b	0	0	z	0
c	a	b	b	0	x	z	x
x	0	0	0	0	x	0	x
y	0	0	x	x	y	0	y
z	0	z	z	0	0	z	0
w	0	0	x	x	y	0	y

Example 4.72. $G790 - G792$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.73. $G794 - G796$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.74. $G798$ is a zero-divisor graph (combining $G166$ and [2, Lemma 3.14] because $b^2 \neq 0$).

G798

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	a	a	0	x	0	a
c	a	a	c	0	0	z	a
x	0	0	0	0	x	0	0
y	0	x	0	x	y	0	x
z	0	0	z	0	0	z	0
w	a	a	a	0	x	0	a

Example 4.75. $G800$ is a zero-divisor graph (combining $G168$ and [2, Lemma 3.14] since $c^2 \neq 0$).

G800

•	a	b	c	x	y	z	w
a	b	b	b	0	0	0	b
b	b	b	b	0	0	0	b
c	b	b	b	0	0	x	b
x	0	0	0	0	0	x	0
y	0	0	0	0	y	y	0
z	0	0	x	x	y	z	x
w	b	b	b	0	0	x	b

Example 4.76. $G801$ is a zero-divisor graph (combining $G170$ and [2, Lemma 3.14] since $b^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	0	a	a	0	0	0	a
	b	a	b	b	0	0	a	b
$G801$	c	a	b	b	0	0	a	b
	x	0	0	0	0	x	0	0
	y	0	0	0	x	y	0	0
	z	0	a	a	0	0	0	a
	w	a	b	b	0	0	a	b

Example 4.77. $G805$ is a zero-divisor graph (combining $G166$ and [2, Lemma 3.14] because $c^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	a	a	a	0	0	0	a
	b	a	a	a	0	x	0	a
$G805$	c	a	a	c	0	0	z	c
	x	0	0	0	0	x	0	0
	y	0	x	0	x	y	0	0
	z	0	0	z	0	0	z	z
	w	a	a	c	0	0	z	c

Example 4.78. $G812$ is a zero-divisor graph (applying [2, Lemma 3.14] to $G168$ because $z^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	b	b	b	0	0	0	0
	b	b	b	b	0	0	0	0
$G812$	c	b	b	b	0	0	x	x
	x	0	0	0	0	0	x	x
	y	0	0	0	0	y	y	y
	z	0	0	x	x	y	z	z
	w	0	0	x	x	y	z	z

Example 4.79. $G813 - G815$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.80. $G832$ is a zero-divisor graph (combining $G169$ and [2, Lemma 3.14] because $b^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	0	x	0	0	0	y	x
	b	x	b	x	x	0	0	b
$G832$	c	0	x	0	0	0	y	x
	x	0	x	0	0	0	0	x
	y	0	0	0	0	0	y	0
	z	y	0	y	0	y	z	0
	w	x	b	x	x	0	0	b

Example 4.81. $G872$ is a zero divisor graph (combining $G119$ and [2, Lemma 3.14] because $y^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	0	a	a	0	0	0	0
	b	a	b	b	0	0	a	0
	c	a	b	c	0	0	a	0
$G872$	x	0	0	0	0	x	0	x
	y	0	0	0	x	y	x	y
	z	0	a	a	0	x	0	x
	w	0	0	0	x	y	x	y

Example 4.82. The connected graph $G884$ which is defined by $N(a) = \{b, c, x, y, z, w\}$, $N(b) = \{a, c, x, y\}$, $N(c) = \{a, b, x, y\}$, $N(x) = \{a, b, c, y\}$, $N(y) = \{a, b, c, x\}$, $N(z) = N(w) = \{a\}$ is a zero-divisor graph because it is a refinement of a star graph..

Example 4.83. $G885$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	0	0	0	0	0	3	4
	2	0	2	0	0	0	2	2
	3	0	0	0	0	0	3	0
$G885$	4	0	0	0	0	0	0	4
	5	0	0	0	0	0	3	4
	6	3	2	3	0	3	6	2
	7	4	2	0	4	4	2	7

Example 4.84. $G886$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.85. $G887$ is a zero-divisor graph (combining $G157$ and [2, Lemma 3.14] because $a^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	a	a	a	0	0	0	a
	b	a	b	a	x	0	0	a
	c	a	a	c	0	y	z	a
$G887$	x	0	x	0	x	0	0	0
	y	0	0	y	0	y	0	0
	z	0	0	z	0	0	z	0
	w	a	a	a	0	0	0	a

Example 4.86. $G888$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.87. $G889$ is a zero-divisor graph.

G889

•	1	2	3	4	5	6	7
1	1	0	7	4	5	4	7
2	0	0	0	0	0	2	0
3	7	0	5	0	0	2	5
4	4	0	0	4	0	4	0
5	5	0	0	0	0	0	0
6	4	2	2	4	0	6	0
7	7	0	5	0	0	0	5

Example 4.88. $G890$ is a zero-divisor graph (combining $G113$ and [2, Lemma 3.14] because $x^2 = 0$).

G890

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	a	a	0	0	x	0
c	a	a	c	0	y	y	0
x	0	0	0	0	0	x	0
y	0	0	y	0	y	y	0
z	0	x	y	x	y	z	x
w	0	0	0	0	0	x	0

Example 4.89. $G891$ is a zero-divisor graph (combining $G137$ and [2, Lemma 3.14] because $a^2 = 0$).

G891

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	0
b	a	b	b	0	0	a	a
c	a	b	c	x	0	a	a
x	0	0	x	x	0	0	0
y	0	0	0	0	y	y	0
z	0	a	a	0	y	y	0
w	0	a	a	0	0	0	0

Example 4.90. $G894$ is a zero-divisor graph.

G894

•	1	2	3	4	5	6	7
1	7	0	3	4	5	2	7
2	0	0	0	0	0	2	0
3	3	0	3	0	0	0	3
4	4	0	0	4	0	0	4
5	5	0	0	0	5	0	5
6	2	2	0	0	0	6	0
7	7	0	3	4	5	0	7

Example 4.91. $G896 - G898$ are zero-divisor graph because they are the refinement of the star graph.

Example 4.92. $G902$ is a zero-divisor graph (combining $G159$ and [2, Lemma 3.14] because $a \times a \neq 0$).

G902

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	b	a	0	0	0	a
c	a	a	c	0	z	z	a
x	0	0	0	x	x	0	0
y	0	0	z	x	x	0	0
z	0	0	z	0	0	0	0
w	a	a	a	0	0	0	a

Example 4.93. $G906$ is a zero-divisor graph.

G906

•	a	b	c	x	y	z	w
a	a	a	0	0	0	0	a
b	a	b	0	x	0	0	a
c	0	0	c	0	0	c	c
x	0	x	0	x	0	0	0
y	0	0	0	0	y	y	y
z	0	0	c	0	y	z	z
w	a	a	c	0	y	z	w

Example 4.94. $G908$ is a zero-divisor graph.

G908

•	a	b	c	x	y	z	w
a	a	a	0	0	0	0	a
b	a	a	0	0	0	0	a
c	0	0	0	0	0	c	c
x	0	0	0	x	x	0	0
y	0	0	0	x	x	c	c
z	0	0	c	0	c	z	z
w	a	a	c	0	c	z	w

Example 4.95. $G909$ is a zero-divisor graph (combining $G159$ and [2, Lemma 3.14] because $y^2 \neq 0$).

G909

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	b	a	0	0	0	0
c	a	a	c	0	z	z	z
x	0	0	0	x	x	0	x
y	0	0	z	x	x	0	x
z	0	0	z	0	0	0	0
w	0	0	z	x	x	0	x

Example 4.96. $G913 - G914$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.97. $G915$ is a zero-divisor graph.

G915

•	1	2	3	4	5	6	7
1	1	0	7	7	0	6	7
2	0	2	0	2	0	0	0
3	7	0	3	7	5	0	7
4	7	2	7	4	0	0	7
5	0	0	5	0	5	0	0
6	6	0	0	0	0	6	0
7	7	0	7	7	0	0	7

Example 4.98. $G916$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.99. $G919 - G925$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.100. $G927$ is a zero-divisor graph.

G927

•	1	2	3	4	5	6	7
1	1	0	4	4	0	1	0
2	0	0	0	0	2	0	0
3	4	0	7	0	2	4	7
4	4	0	0	0	0	4	0
5	0	2	2	0	5	2	0
6	1	0	4	4	2	1	0
7	0	0	7	0	0	0	7

Example 4.101. $G929$ is a zero-divisor graph (combining $G166$ and [2, Lemma 3.14] since $a^2 \neq 0$).

G929

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	a	a	0	x	0	a
c	a	a	c	0	0	z	a
x	0	0	0	0	x	0	0
y	0	x	0	x	y	0	0
z	0	0	z	0	0	z	0
w	a	a	a	0	0	0	a

Example 4.102. $G930$ is a zero-divisor graph (combining $G118$ and [2, Lemma 3.14] since $x^2 = 0$).

G930

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	0
b	a	b	b	0	0	a	0
c	a	b	b	0	x	a	0
x	0	0	0	0	x	0	0
y	0	0	x	x	y	x	x
z	0	a	a	0	x	0	0
w	0	0	0	0	x	0	0

Example 4.103. G_{932} is a zero-divisor graph (combining G_{185} and [2, Lemma 3.14] since $c^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	a	a	0	0	0	0	0
	b	a	a	x	0	0	0	x
G932	c	0	x	c	x	0	x	c
	x	0	0	x	0	0	0	x
	y	0	0	0	0	y	y	0
	z	0	0	x	0	y	y	x
	w	0	x	c	x	0	x	c

Example 4.104. G_{934} is a zero-divisor graph (combining G_{186} and [2, Lemma 3.14] because $c^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	a	a	a	0	0	0	a
	b	a	a	a	0	0	0	a
G934	c	a	a	c	x	0	0	c
	x	0	0	x	x	0	0	x
	y	0	0	0	0	y	y	0
	z	0	0	0	0	y	y	0
	w	a	a	c	x	0	0	c

Example 4.105. G_{939} is a zero-divisor graph (combining G_{137} and [2, Lemma 3.14] because $y^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	0	a	a	0	0	0	0
	b	a	b	b	0	0	a	0
G939	c	a	b	c	x	0	a	0
	x	0	0	x	x	0	0	0
	y	0	0	0	0	y	y	y
	z	0	a	a	0	y	y	y
	w	0	0	0	0	y	y	y

Example 4.106. G_{944} is a zero-divisor graph (combining G_{185} and [2, Lemma 3.14] because $b^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	0	a	0	0	0	0	a
	b	a	b	a	0	0	a	b
G944	c	0	a	0	0	0	0	a
	x	0	0	0	0	x	0	0
	y	0	0	0	x	y	x	0
	z	0	a	0	0	x	0	a
	w	a	b	a	0	0	a	b

Example 4.107. G_{948} is a zero-divisor graph (combining G_{140} and [2, Lemma 3.14] because $x^2 \neq 0$).

G948

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	b	a	0	0	0	0
c	a	a	c	0	z	z	0
x	0	0	0	x	x	0	x
y	0	0	z	x	y	z	x
z	0	0	z	0	z	z	0
w	0	0	0	x	0	0	x

Example 4.108. $G950 - G952$ are zero-divisor graphs because they are the refinement of star graphs.

Example 4.109. $G957$ is a zero-divisor graph (combining $G168$ and [2, Lemma 3.14] since $a^2 \neq 0$).

G957

•	a	b	c	x	y	z	w
a	b	b	b	0	0	0	b
b	b	b	b	0	0	0	b
c	b	b	b	0	0	x	b
x	0	0	0	0	0	x	0
y	0	0	0	0	y	y	0
z	0	0	x	x	y	z	0
w	b	b	b	0	0	0	b

Example 4.110. $G972$ is a zero-divisor graph.

G972

•	1	2	3	4	5	6	7
1	0	0	1	0	0	0	0
2	0	2	0	4	5	5	4
3	1	0	3	0	0	1	1
4	0	4	0	4	0	0	4
5	0	5	0	0	5	5	0
6	0	5	1	0	5	5	0
7	0	4	1	4	0	0	4

Example 4.111. $G975$ is a zero-divisor graph (combining $G166$ and [2, Lemma 3.14] since $y^2 \neq 0$).

G975

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	a	a	0	x	0	x
c	a	a	c	0	0	z	0
x	0	0	0	0	x	0	x
y	0	x	0	x	y	0	y
z	0	0	z	0	0	z	0
w	0	x	0	x	y	0	y

Example 4.112. The connected graph $G1007$ which is defined by $N(a) = N(b) = N(c) = \{d, e, f, g\}$, $N(d) = N(e) = N(f) = N(g) = \{a, b, c\}$ is a zero-divisor graph because it is a bipartite graph.

Example 4.113. $G1008 - G1009$ are zero-divisor graph because they are the refinement of the star graph.

Example 4.114. $G1012$ is a zero-divisor graph.

Example 4.115. $G1013 - G1015$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.116. $G1016$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	3	0	0	1
	2	0	2	0	2	0	0	0
G1016	3	3	0	0	0	0	0	3
	4	3	2	0	2	0	0	3
	5	0	0	0	0	0	0	5
	6	0	0	0	0	0	0	5
	7	1	0	3	3	5	5	7

Example 4.117. $G1017$ is a zero-divisor graph (combining $G178$ and [2, Lemma 3.14] because $y^2 \neq 0$).

	•	a	b	c	x	y	z	w
	a	0	a	0	0	0	0	0
	b	a	b	a	0	a	a	a
G1017	c	0	a	0	0	0	0	0
	x	0	0	0	x	x	0	x
	y	0	a	0	x	x	0	x
	z	0	a	0	0	0	0	0
	w	0	a	0	x	x	0	x

Example 4.118. $G1018$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.119. $G1019$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	4	7	6	7
	2	0	2	0	0	2	0	0
G1019	3	3	0	3	0	0	3	0
	4	4	0	0	4	0	4	0
	5	7	2	0	0	2	0	0
	6	6	0	3	4	0	6	0
	7	7	0	0	0	0	0	0

Example 4.120. $G1020$ is a zero-divisor graph (combining $G157$ and [2, Lemma 3.14] because $y^2 \neq 0$).

G1020

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	b	a	x	0	0	0
c	a	a	c	0	y	z	y
x	0	x	0	x	0	0	0
y	0	0	y	0	y	0	y
z	0	0	z	0	0	z	0
w	0	0	y	0	y	0	y

Example 4.121. $G1025$ – $G1028$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.122. $G1029$ is a zero-divisor graph.

G1029

•	1	2	3	4	5	6	7
1	1	0	1	0	0	0	0
2	0	2	0	0	0	2	2
3	1	0	1	0	0	5	4
4	0	0	0	0	0	0	4
5	0	0	0	0	0	5	0
6	0	2	5	0	5	6	2
7	0	2	4	4	0	2	7

Example 4.123. $G1031$ – $G1032$ are zero-divisor graphs because they are the refinement of a star graph.

Example 4.124. $G1035$ is a zero-divisor graph (combining $G157$ and [2, Lemma 3.14] since $x^2 \neq 0$).

G1035

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	b	a	x	0	0	x
c	a	a	c	0	y	z	0
x	0	x	0	x	0	0	x
y	0	0	y	0	y	0	0
z	0	0	z	0	0	z	0
w	0	x	0	x	0	0	x

Example 4.125. $G1036$ is a zero-divisor graph.

G1036

•	1	2	3	4	5	6	7
1	0	0	0	0	0	4	3
2	0	0	0	0	0	4	3
3	0	0	0	0	0	0	3
4	0	0	0	0	0	4	0
5	0	0	0	0	0	4	3
6	4	4	0	4	4	6	0
7	3	3	3	0	3	0	7

Example 4.126. $G1037$ is a zero-divisor graph.

G1037

•	1	2	3	4	5	6	7
1	1	1	0	0	5	6	7
2	1	1	0	0	5	6	7
3	0	0	3	4	0	0	0
4	0	0	4	0	0	0	0
5	5	5	0	0	5	0	0
6	6	6	0	0	0	6	0
7	7	7	0	0	0	0	7

Example 4.127. $G1038-G1042$ are zero-divisor graphs because they are the refinement of \star graph.

Example 4.128. $G1045$ is a zero-divisor graph.

G1045

•	1	2	3	4	5	6	7
1	1	0	3	0	5	3	3
2	0	2	0	2	0	0	0
3	3	0	3	0	0	3	3
4	0	2	0	2	0	0	0
5	5	0	0	0	5	0	0
6	3	0	3	0	0	3	3
7	3	0	3	0	0	3	3

Example 4.129. $G1046-G1049$ are zero-divisor graphs because they are the refinement of the star graph with seven vertices.

Example 4.130. $G1050$ is a zero-divisor graph.

G1050

•	1	2	3	4	5	6	7
1	1	0	6	6	0	6	0
2	0	2	0	2	0	0	2
3	6	0	0	0	0	0	5
4	6	2	0	2	0	0	2
5	0	0	0	0	0	0	5
6	6	0	0	0	0	0	0
7	0	2	5	2	5	0	7

Example 4.131. $G1052$ is a zero-divisor graph.

G1052

•	1	2	3	4	5	6	7
1	1	2	3	6	0	6	0
2	2	2	0	6	0	6	0
3	3	0	3	0	0	0	0
4	6	6	0	0	0	0	5
5	0	0	0	0	0	0	5
6	6	6	0	0	0	0	0
7	0	0	0	5	5	0	7

Example 4.132. $G1053$ is a zero-divisor graph.

G1053

•	1	2	3	4	5	6	7
1	0	0	0	0	0	1	0
2	0	2	0	4	4	0	2
3	0	0	3	0	3	0	0
4	0	4	0	0	0	0	4
5	0	4	3	0	3	0	4
6	1	0	0	0	0	6	1
7	0	2	0	4	4	1	2

Example 4.133. G_{1056} is a zero-divisor graph.

G1056

•	1	2	3	4	5	6	7
1	1	0	0	4	5	5	4
2	0	2	0	0	0	0	2
3	0	0	3	0	0	3	0
4	4	0	0	4	0	0	4
5	5	0	0	0	5	5	0
6	5	0	3	0	5	6	0
7	4	2	0	4	0	0	7

Example 4.134. G_{1057} is a zero-divisor graph (combining G_{185} and [2, Lemma 3.14] because of $b^2 \neq 0$).

G1057

•	a	b	c	x	y	z	w
a	a	a	0	0	0	0	a
b	a	a	x	0	0	0	a
c	0	x	c	x	0	x	x
x	0	0	x	0	0	0	0
y	0	0	0	0	y	y	0
z	0	0	x	0	y	y	0
w	a	a	x	0	0	0	a

Example 4.135. G_{1059} is a zero-divisor graph.

G1059

•	1	2	3	4	5	6	7
1	1	0	0	0	0	0	1
2	0	2	0	4	4	2	0
3	0	0	3	0	3	3	0
4	0	4	0	0	0	4	0
5	0	4	3	0	3	5	0
6	0	2	3	4	5	6	0
7	1	0	0	0	0	0	1

Example 4.136. G_{1062} is a zero-divisor graph (combining G_{141} and [2, Lemma 3.14] since $x^2 = 0$).

G1062

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	b	b	0	0	z	0
c	a	b	b	0	x	z	0
x	0	0	0	0	x	0	0
y	0	0	x	x	y	0	x
z	0	z	z	0	0	z	0
w	0	0	0	0	x	0	0

Example 4.137. G_{1067} is a zero-divisor graph (combining G_{166} and [2, lemma 3.14] since $z^2 \neq 0$).

G1067

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	a	a	0	x	0	0
c	a	a	c	0	0	z	z
x	0	0	0	0	x	0	0
y	0	x	0	x	y	0	0
z	0	0	z	0	0	z	z
w	0	0	z	0	0	z	z

Example 4.138. G_{1072} is a zero-divisor graph (combining G_{159} and [2, lemma 3.14] since $x^2 \neq 0$).

G1072

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	0
b	a	b	a	0	0	0	0
c	a	a	c	0	z	z	0
x	0	0	0	x	x	0	x
y	0	0	z	x	x	0	x
z	0	0	z	0	0	0	0
w	0	0	0	x	x	0	x

Example 4.139. $G_{1077} - G_{1081}$ are the zero-divisor graphs since they are the refinement of a star graph with seven vertices.

Example 4.140. G_{1085} is a zero-divisor graph (combining G_{119} and [2, Lemma 3.14] since $x^2 = 0$).

G1085

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	0
b	a	b	b	0	0	a	0
c	a	b	c	0	0	a	0
x	0	0	0	0	x	0	0
y	0	0	0	x	y	x	x
z	0	a	a	0	x	0	0
w	0	0	0	0	x	0	0

Example 4.141. G_{1088} is a zero-divisor graph (applying [2] to G_{189} since $a^2 \neq 0$).

G1088

•	a	b	c	x	y	z	w
a	a	a	a	0	0	0	a
b	a	a	a	0	0	0	a
c	a	a	a	0	0	0	a
x	0	0	0	0	x	0	0
y	0	0	0	x	y	z	0
z	0	0	0	0	z	z	0
w	a	a	a	0	0	0	a

Example 4.142. $G1106$ is a zero-divisor graph (combining $G140$ and [2, Lemma 3.14] since $y^2 \neq 0$).

G1106

•	a	b	c	x	y	z	w
a	0	a	a	0	0	0	0
b	a	b	b	0	0	a	0
c	a	b	b	0	0	a	0
x	0	0	0	0	x	0	x
y	0	0	0	x	y	0	y
z	0	a	a	0	0	0	0
w	0	0	0	x	y	0	y

Example 4.143. $G1108$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.144. $G1109$ is a zero-divisor graph (combining $G157$ and [2, Lemma 3.14] since $y^2 = 0$).

G1109

•	1	2	3	4	5	6	7
1	1	0	3	4	3	6	6
2	0	2	0	0	2	0	0
3	3	0	3	0	3	0	0
4	4	0	0	4	0	0	0
5	3	2	3	0	5	0	0
6	6	0	0	0	0	0	0
7	6	0	0	0	0	0	0

Example 4.145. $G1110$ is a zero-divisor graph because it is a refinement of a star graph.

Example 4.146. $G1111$ is a zero-divisor graph.

G1111

•	1	2	3	4	5	6	7
1	1	0	3	3	3	6	6
2	0	2	0	2	0	0	0
3	3	0	0	0	0	0	0
4	3	2	0	2	0	0	0
5	3	0	0	0	0	0	0
6	6	0	0	0	0	3	3
7	6	0	0	0	0	3	3

Example 4.147. G_{1113} – G_{1114} are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.148. G_{1115} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	1	1	0	0	0	0
	2	1	2	1	4	0	0	4
G_{1115}	3	1	1	3	0	5	6	0
	4	0	4	0	0	0	0	0
	5	0	0	5	0	5	0	0
	6	0	0	6	0	0	6	0
	7	0	4	0	0	0	0	0

Example 4.149. G_{1116} – G_{1119} are zero-divisor graphs because they are the refinement a star graph with seven vertices.

Example 4.150. G_{1121} – G_{1124} are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.151. G_{1125} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	0	0	0	1
	2	0	2	6	4	0	6	0
G_{1125}	3	0	6	3	0	5	6	5
	4	0	4	0	4	0	0	0
	5	0	0	5	0	0	0	0
	6	0	6	6	0	0	6	0
	7	1	0	5	0	0	0	1

Example 4.152. G_{1126} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	3	0	3	2	0	0	3
	2	0	0	0	2	0	0	0
G_{1126}	3	3	0	3	0	0	0	3
	4	2	2	0	4	0	0	0
	5	0	0	0	0	0	0	5
	6	0	0	0	0	0	0	5
	7	3	0	3	0	5	5	7

Example 4.153. G_{1128} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	2	3	0	0	7	7
	2	2	0	0	0	0	3	3
G_{1128}	3	3	0	0	0	0	0	0
	4	0	0	0	4	4	0	0
	5	0	0	0	4	4	0	0
	6	7	3	0	0	0	0	0
	7	7	3	0	0	0	0	0

Example 4.154. G_{1129} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	0	3	3	0
	2	0	2	0	4	4	4	4
G_{1129}	3	3	0	0	0	0	0	0
	4	0	4	0	0	0	0	0
	5	3	4	0	0	0	0	0
	6	3	4	0	0	0	0	0
	7	0	4	0	0	0	0	0

Example 4.155. G_{1131} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	4	0	0	4	4	2	0
	2	0	0	0	0	0	2	0
G_{1131}	3	0	0	0	0	0	2	0
	4	4	0	0	4	4	0	0
	5	4	0	0	4	4	0	0
	6	2	2	2	0	0	6	2
	7	0	0	0	0	0	2	0

Example 4.156. G_{1132} is a zero-divisor graph because it is a refinement of star graph with six vertices.

Example 4.157. G_{1134} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	4	4	0	6	6
	2	0	0	0	0	2	0	0
G_{1134}	3	4	0	0	0	2	0	0
	4	4	0	0	0	0	0	0
	5	0	2	2	0	5	0	0
	6	6	0	0	0	0	6	6
	7	6	0	0	0	0	6	6

Example 4.158. G_{1135} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	4	0	0	4	4	3	2
	2	0	0	0	0	0	0	2
G_{1135}	3	0	0	0	0	0	3	0
	4	4	0	0	4	4	0	0
	5	4	0	0	4	4	0	0
	6	3	0	3	0	0	6	0
	7	2	2	0	0	0	0	7

Example 4.159. G_{1137} is a zero-divisor graph.

G1137

•	1	2	3	4	5	6	7
1	1	0	0	1	0	0	1
2	0	2	0	2	0	2	0
3	0	0	3	0	3	0	0
4	1	2	0	4	0	2	1
5	0	0	3	0	3	0	0
6	0	2	0	2	0	2	0
7	1	0	0	1	0	0	1

Example 4.160. $G1138$ – $G1144$ are zero-divisor graphs because they are the refinement of a star graphs with seven vertices.

Example 4.161. $G1145$ is a zero-divisor graph.

G1145

•	1	2	3	4	5	6	7
1	0	0	0	0	0	0	1
2	0	0	0	0	0	2	0
3	0	0	5	0	5	2	1
4	0	0	0	0	0	2	1
5	0	0	5	0	5	0	0
6	0	2	2	2	0	6	0
7	1	0	1	1	0	0	7

Example 4.162. $G1147$ is a zero-divisor graph.

G1147

•	1	2	3	4	5	6	7
1	1	0	7	7	0	0	7
2	0	2	5	0	5	6	0
3	7	5	5	0	5	0	0
4	7	0	0	0	0	0	0
5	0	5	5	0	5	0	0
6	0	6	0	0	0	6	0
7	7	0	0	0	0	0	0

Example 4.163. $G1148$ is a zero-divisor graph.

G1148

•	1	2	3	4	5	6	7
1	1	1	0	0	0	0	0
2	1	1	0	0	0	0	0
3	0	0	0	0	0	3	0
4	0	0	0	4	4	0	4
5	0	0	0	4	4	3	4
6	0	0	3	0	3	6	3
7	0	0	0	4	4	3	4

Example 4.164. $G1149$ – $G1152$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.165. $G1157$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	2	0	0	3
	2	0	0	0	2	0	0	0
G1157	3	3	0	0	0	0	0	0
	4	2	2	0	4	0	0	0
	5	0	0	0	0	5	5	5
	6	0	0	0	0	5	5	5
	7	3	0	0	0	5	5	5

Example 4.166. $G1163$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	0	1	0	0
	2	0	2	0	0	2	0	0
G1163	3	0	0	3	0	0	3	3
	4	0	0	0	4	0	4	4
	5	1	2	0	0	5	0	0
	6	0	0	3	4	0	6	6
	7	0	0	3	4	0	6	6

Example 4.167. $G1169$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	5	0	5	6	0
	2	0	2	0	2	0	0	2
G1169	3	5	0	5	0	5	0	0
	4	0	2	0	2	0	0	2
	5	5	0	5	0	5	0	0
	6	6	0	0	0	0	6	0
	7	0	2	0	2	0	0	2

Example 4.168. $G1173$ is a zero-divisor graph because it is a refinement of a star graph with seven vertices.

Example 4.169. $G1174$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	0	0	0	0	0	0	1
	2	0	5	0	0	5	0	1
G1174	3	0	0	0	0	0	0	1
	4	0	0	0	0	0	0	1
	5	0	5	0	0	5	0	0
	6	0	0	0	0	0	0	1
	7	1	1	1	1	0	1	7

Example 4.170. $G1175$ – $G1176$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.171. $G1178$ – $G1180$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.172. G_{1181} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	4	4	4	4
	2	0	2	0	0	0	0	2
G1181	3	0	0	3	0	0	3	0
	4	4	0	0	0	0	0	0
	5	4	0	0	0	0	0	0
	6	4	0	3	0	0	3	0
	7	4	2	0	0	0	0	2

Example 4.173. G_{1182} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	3	3	3	0
	2	0	2	0	0	7	0	7
G1182	3	3	0	0	0	0	0	0
	4	3	0	0	0	0	0	0
	5	3	7	0	0	0	0	0
	6	3	0	0	0	0	0	0
	7	0	7	0	0	0	0	0

Example 4.174. G_{1183} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	2	0	0	5	6	2
	2	2	2	0	0	0	0	2
G1183	3	0	0	0	3	0	0	0
	4	0	0	3	4	0	0	0
	5	5	0	0	0	5	0	0
	6	6	0	0	0	0	6	0
	7	2	2	0	0	0	0	2

Example 4.175. G_{1184} – G_{1193} are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.176. G_{1194} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	3	3	0	3	0
	2	0	2	0	0	0	2	0
G1194	3	3	0	0	0	0	0	0
	4	3	0	0	0	0	0	0
	5	0	0	0	0	5	7	7
	6	3	2	0	0	7	2	0
	7	0	0	0	0	7	0	0

Example 4.177. G_{1195} is a zero-divisor graph.

G1195

•	1	2	3	4	5	6	7
1	1	0	0	0	0	0	1
2	0	2	0	0	2	2	0
3	0	0	3	0	0	3	0
4	0	0	0	4	4	0	0
5	0	2	0	4	5	2	0
6	0	2	3	0	2	6	0
7	1	0	0	0	0	0	1

Example 4.178. $G1196$ is a zero-divisor graph.

G1196

•	1	2	3	4	5	6	7
1	0	0	0	0	0	4	3
2	0	0	0	0	0	0	3
3	0	0	0	0	0	0	3
4	0	0	0	0	0	4	0
5	0	0	0	0	0	4	0
6	4	0	0	4	4	6	0
7	3	3	3	0	0	0	7

Example 4.179. $G1197$ – $G1199$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.180. $G1200$ is a zero-divisor graph.

G1200

•	1	2	3	4	5	6	7
1	1	0	0	5	5	0	5
2	0	0	0	0	0	2	0
3	0	0	3	0	0	0	3
4	5	0	0	0	0	2	0
5	5	0	0	0	0	0	0
6	0	2	0	2	0	6	0
7	5	0	3	0	0	0	3

Example 4.181. $G1202$ is a zero-divisor graph.

G1202

•	1	2	3	4	5	6	7
1	3	0	3	3	2	0	0
2	0	0	0	0	2	0	0
3	3	0	3	3	0	0	0
4	3	0	3	3	0	0	0
5	2	2	0	0	5	0	0
6	0	0	0	0	0	6	6
7	0	0	0	0	0	6	6

Example 4.182. $G1203$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	0	1	0	0
	2	0	2	0	0	0	6	6
G1203	3	0	0	3	3	3	0	0
	4	0	0	3	3	3	0	0
	5	1	0	3	3	5	0	0
	6	0	6	0	0	0	0	0
	7	0	6	0	0	0	0	0

Example 4.183. G_{1205} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	0	0	6	6
	2	0	2	0	0	5	0	5
G1205	3	0	0	3	4	0	0	4
	4	0	0	4	0	0	0	0
	5	0	5	0	0	0	0	0
	6	6	0	0	0	0	0	0
	7	6	5	4	0	0	0	0

Example 4.184. G_{1206} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	0	5	5	5
	2	0	2	2	2	0	0	0
G1206	3	0	2	2	2	0	0	0
	4	0	2	2	2	0	0	0
	5	5	0	0	0	0	0	0
	6	5	0	0	0	0	0	0
	7	5	0	0	0	0	0	0

Example 4.185. $G_{1207} - 1208$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.186. G_{1210} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	1	0	0	0	0	1
	2	1	1	0	0	0	0	1
G1210	3	0	0	3	6	0	6	0
	4	0	0	6	5	5	0	0
	5	0	0	0	5	5	0	0
	6	0	0	6	0	0	0	0
	7	1	1	0	0	0	0	1

Example 4.187. $G_{1213} - G_{1224}$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.188. G_{1225} is a zero-divisor graph.

G1225

•	1	2	3	4	5	6	7
1	1	0	0	0	6	6	6
2	0	2	0	0	2	0	0
3	0	0	3	3	0	0	0
4	0	0	3	3	0	0	0
5	6	2	0	0	2	0	0
6	6	0	0	0	0	0	0
7	6	0	0	0	0	0	0

Example 4.189. G_{1226} is a zero-divisor graph.

G1226

•	1	2	3	4	5	6	7
1	0	0	0	1	0	0	0
2	0	0	0	1	0	0	0
3	0	0	3	0	5	5	5
4	1	1	0	4	0	0	0
5	0	0	5	0	0	0	0
6	0	0	5	0	0	0	0
7	0	0	5	0	0	0	0

Example 4.190. G_{1227} – G_{1230} are zero-divisor graphs because they are the refinement of star graph with seven vertices.

Example 4.191. G_{1231} is a zero-divisor graph.

G1231

•	1	2	3	4	5	6	7
1	1	0	0	0	7	0	7
2	0	2	0	0	6	6	0
3	0	0	3	3	0	0	0
4	0	0	3	3	0	0	0
5	7	6	0	0	0	0	0
6	0	6	0	0	0	0	0
7	7	0	0	0	0	0	0

Example 4.192. G_{1232} is a zero-divisor graph.

G1232

•	1	2	3	4	5	6	7
1	0	0	0	0	1	0	0
2	0	2	0	2	0	0	0
3	0	0	3	0	0	6	6
4	0	2	0	2	1	0	0
5	1	0	0	1	5	0	0
6	0	0	6	0	0	0	0
7	0	0	6	0	0	0	0

Example 4.193. G_{1233} is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	0	0	0	1
	2	0	2	0	0	0	2	0
G1233	3	0	0	3	3	3	0	0
	4	0	0	3	3	3	0	0
	5	0	0	3	3	3	0	0
	6	0	2	0	0	0	2	0
	7	1	0	0	0	0	0	1

Example 4.194. $G1234$ – $G1242$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

Example 4.195. $G1243$ is a zero-divisor graph.

	•	1	2	3	4	5	6	7
	1	1	0	0	0	0	0	1
	2	0	2	0	0	0	2	0
G1243	3	0	0	0	0	3	0	0
	4	0	0	0	0	3	0	0
	5	0	0	3	3	5	0	0
	6	0	2	0	0	0	2	0
	7	1	0	0	0	0	0	1

Example 4.196. $G1244$ – $G1252$ are zero-divisor graphs because they are the refinement of a star graph with seven vertices.

5. NON ZERO-DIVISOR GRAPH WITH SEVEN VERTICES THAT SATISFIES THE GIVEN CONDITION \star

5.1. a non-zero divisor graph with seven vertices produced by emanating an end from the highest degree vertex of a zero-divisor graph with six vertices produces. By Lemma 3.3, we know that emanating an end from the highest degree vertex of a zero-divisor graph with six vertices always produces a connected graph satisfies the condition \star .

Theorem 5.1. (1) $G184 : N(a) = \{c, x, y, z\}, N(b) = \{x, y, z\}, N(c) = \{a, y\}, N(x) = \{a, b, y, z\}, N(y) = \{a, b, c, x\}, N(z) = \{a, b, x\}$
 (2) $G194 : N(a) = N(b) = \{c, x, y, z\}, N(c) = \{a, b, x, y\}, N(x) = \{a, b, c, y\}, N(y) = \{a, b, c, x\}, N(z) = \{a, b\}$
 (3) $G204 : N(a) = N(b) = \{c, x, y, z\}, N(c) = N(z) = \{a, b, x, y\}, N(x) = \{a, b, c, z\}, N(y) = \{a, b, c, z\}$

are zero-divisor graphs which were proved in 24 exceptional cases in [5]. By adding a vertex w and an edge xw , the resulting graph is not a zero-divisor graph.

Proof. The proof is given in the following Examples 5.19, 5.25, and 5.34. □

5.2. Emanating an end from the highest degree vertex of a graph with six vertices which satisfies the condition \star but is not a zero-divisor graph.

Theorem 5.2. [2] *Up to isomorphism, there are four connected graphs with six vertices that satisfy condition \star but are not the zero-divisor graphs. They are*

- (1) G98 : $N(a) = \{b, d\}$, $N(b) = \{a, c, e\}$, $N(c) = \{b\}$, $N(d) = \{a, e\}$, $N(e) = \{b, d, f\}$, $N(f) = \{e\}$.
- (2) G145 : $N(a) = \{b, d\}$, $N(b) = \{a, c, e, f\}$, $N(c) = N(f) = \{b, e\}$, $N(d) = \{a, e\}$, $N(e) = \{b, c, d, f\}$.
- (3) G163 : $N(a) = \{b, f\}$, $N(b) = \{a, c, d, f\}$, $N(c) = \{b, d\}$, $N(d) = \{b, c, e, f\}$, $N(e) = \{d, f\}$, $N(f) = \{a, b, d, e\}$.
- (4) G181 : $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c, e\}$, $N(e) = \{b, c, d, f\}$, $N(f) = \{a, b, c, e\}$.

Theorem 5.3. *By adding a vertex and an edge to the vertex with maximum degree of the above graphs in Theorem 5.2, we will not get any zero-divisor graphs.*

Proof. The proof will be given in Examples 5.1, 5.7, 5.11, and 5.17. \square

Lemma 5.1. *Let G be a connected graph. Let $x \in V(G)$ be a vertex with the highest degree. Suppose there are no ends emanating from x in G . Let H be a graph defined by emanating an end from vertex x . If G is not a zero divisor graph but satisfies the necessary condition \star , then H is not a zero-divisor graph.*

Proof. Notice $N(w) = \{x\}$. Suppose a and b are two dis-connected vertices in G . If one of a or b is not an end in G , then $|N(a) \cup N(b)| \geq 2$. It follows that $N(a) \cup N(b)$ is not a subset of $\overline{N(w)}$. Hence $ab \neq w$. If $\deg(a) = \deg(b) = 1$, then a and b are the ends in G . By the definition of G , a and b do not emanates from x . It follows that $N(a) \cup N(b)$ is not a subset of $\overline{N(w)}$. Hence $ab \neq w$.

Suppose H is a zero-divisor graph. Then G must be a zero-divisor graph. This is a contradiction! \square

Lemma 5.2. *Let G be a connected graph which does not satisfies the necessary condition \star of a zero-divisor graph. Let x be a vertex of G with highest degree. Let H be a graph defined by adding a new vertex w and an new edge xw to G . Then H is not a zero-divisor graph.*

Proof. Suppose a and b be two dis-connected vertices in G such that $N(a) \cup N(b)$ is not subset of any $\overline{N(y)}$ in G . Then it is easy to see that $N(a) \cup N(b)$ is not subset of any $\overline{N(y)}$ in H .

Hence H is not a zero-divisor graph. \square

5.3. Non zero-divisor graphs with seven vertices that satisfy the condition \star produced by applying [2, Lemma 3.14] to any graphs in Theorem 5.2.

Theorem 5.4. *One can not produce any zero-divisor graphs if we apply [2, Lemma 3.14] to any graphs in Theorem 5.2.*

Proof. The proof will be given in Examples 5.1, 5.3, 5.13, 5.14, 5.15, 5.21, 5.29, 5.38, 5.39, and 5.42. \square

Lemma 5.3. *If G is graph that violates the necessary condition \star of a zero-divisor graph, then a graph H formed by the method of [2, Lemma 3.14] still violates the necessary condition \star of a zero-divisor graph.*

Proof. Suppose $N_G(a) \cup N_G(b)$ is not subset of any $\overline{N_G(x)}$ in old graph. Suppose there exists a vertex $y \notin V(H) - V(G)$ such that $N_H(a) \cup N_H(b) \subset \overline{N_H(y)}$. Notice $N_H(a) \cap V(G) = N_G(a)$, $N_H(b) \cap V(G) = N_G(b)$, $N_H(y) \cap V(G) = N_G(y)$. We get a contradiction. If $y \in V(H) - V(G)$, notice $N_H(y) \cap V(G) = N(x)$. This again is a contradiction. \square

5.4. satisfying \star condition but not zero-divisor graph.

Theorem 5.5. *The following are all the connected graphs in [4] which satisfy the \star condition but are not zero-divisor graphs.*

$G322, G383, G405, G475, G482, G490, G504, G600, G602, G607,$
 $G617, G627, G635, G669, G677, G742, G750, G754, G766, G772, G793, G799, G803, G808,$
 $G893, G899, G907, G917 - G918, G928, G933, G938, G953, G1024, G1030, G1034,$
 $G1043 - G1044, G1060, G1083, G1120, G1130, G1146, G1177,$

Proof. The proof is given in the following examples in this section. \square

Example 5.1. The connected graph $G322$ with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{b, e\}$, $N(b) = \{a, c, d, f\}$, $N(c) = \{b\}$, $N(d) = \{b\}$, $N(e) = \{a, f\}$, $N(f) = \{b, e, g\}$, $N(g) = \{f\}$ is not a zero-divisor graph. If we remove d , we get graph (1) in Theorem 5.2. We have the following multiplication table.

\bullet	a	b	c	d	e	f	g
a		0			0	f	f
b	0	0	0	0	b	0	b
c		0			b	f	
d		0			b	f	
e	0	b	b	b		0	
f	f	0	f	f	0	0	0
g	f	b				0	

By Light's Associativity Test, we have the following a -table.

a	a^2	0	ac	ad	0	f	f
a^2		0		0	0	f	f
0	0	0	0	0	0	0	0
ac		0		0	0	f	f
ad		0		0	0	f	f
0	0	0	0	0	0	0	0
f	f	0	f	f	0	0	0
f	f	0	f	f	0	0	0

By checking each spectrum of vertices of G , we get $a^2 = a$, $ac = a$, $ad = a$. Hence we have the following multiplication Table.

•	a	b	c	d	e	f	g
a	a	0	a	a	0	f	f
b	0	0	0	0	b	0	b
c	a	0			b	f	
d	a	0			b	f	
e	0	b	b	b	e	0	e
f	f	0	f	f	0	0	0
g	f	b			e	0	

By Light's Associativity Test, we have the following d -table.

d	a	0	cd	d^2	b	f	dg
a	a	0	a	a	0	f	f
0	0	0	0	0	0	0	0
cd	a	0			0	f	
d^2	a	0			0	f	
b	0	0	0	0	b	0	b
f	f	0	f	f	0	0	0
dg	f	0			b	0	

Since $dg \in \{b, f\}$, we get a contradiction.

Second proof: Notice this is a complete bipartite graph together with three ends emanating from two distinct vertices c and x . By [5, Theorem IV.5], we conclude this graph is not a zero-divisor graph.

Example 5.2. $G383$ satisfies the necessary condition \star but is not a zero-divisor graph. $G383$ can be defined by $N(1) = \{3, 4\}$, $N(2) = \{3\}$, $N(3) = \{1, 2, 4, 6\}$, $N(4) = \{1, 3, 5, 6\}$, $N(5) = \{4\}$, $N(6) = \{3, 4, 7\}$, $N(7) = \{6\}$.

It follows that

$$N(2) \cup N(4) = N(4) \cup N(7) = \{1, 3, 5, 6\} = N(4)$$

and

$$N(3) \cup N(5) = N(3) \cup N(7) = \{1, 2, 4, 6\} = N(3)$$

•	1	2	3	4	5	6	7
1			0	0			
2			0	4			
3	0	0		0	3	0	3
4	0	4	0		0	0	4
5			3	0			
6			0	0			0
7			3	4		0	

By checking 7- table, we get $5 \bullet 7 = 3$. It follows that $1 \bullet 7 = 4$. But then we can't define $2 \bullet 7$.

Example 5.3. The connected graph $G405$ with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{c, d, e\}$, $N(b) = \{c, d, e, g\}$, $N(c) = \{a, b\}$, $N(d) = \{a, b, f\}$, $N(e) = \{a, b\}$, $N(f) = \{d\}$, $N(g) = \{b\}$ is not a zero-divisor graph. If we remove e , we get graph (1) in Theorem 5.2.

We have the following multiplication Table.

•	a	b	c	d	e	f	g
a			0	0	0		
b			0	0	0		0
c	0	0					
d	0	0				0	
e	0	0					
f				0			
g		0					

Since

$$N(a) \cup N(b) = \{c, d, e, g\} = N(b), N(a) \cup N(g) \subset \overline{N(b)}$$

$$N(b) \cup N(f) = N(b), N(c) \cup N(d) = N(d), N(c) \cup N(f) \subset \overline{N(d)}$$

$$N(d) \cup N(e) = N(d), N(e) \cup N(f) \subset \overline{N(d)}, N(d) \cup N(g) = N(d)$$

we get the following updated Multiplication-table.

•	a	b	c	d	e	f	g
a		b	0	0	0		b
b	b	0	0	0	0	b	0
c	0	0		d		d	
d	0	0	d	0	d	0	d
e	0	0		d		d	
f		b	d	0	d		
g	b	0		d			

Let us check the a -table.

a	a^2	b	0	0	0	af	b
a^2		b	0	0	0		b
b	b	0	0	0	0	b	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
af		b	0	0			b
b	b	0	0	0	0	b	0

we get $a^2 = a$, $af = a$. We have the following updated multiplication table.

•	a	b	c	d	e	f	g
a	a	b	0	0	0	a	b
b	b	0	0	0	0	b	0
c	0	0		d		d	
d	0	0	d	0	d	0	d
e	0	0		d		d	
f	a	b	d	0	d		
g	b	0		d			

Let's check the f -table.

f	a	b	d	0	d	f^2	fg
a	a	b	0	0	0	a	b
b	b	0	0	0	0	b	0
d	0	0	d	0	d	0	d
0	0	0	0	0	0	0	0
d	0	0	d	0	d	0	d
f^2	a	b	0	0	0		
fg	b	0	d	0	d		

Then we can not define fg since in the Multiplication-table, no spectrum of a, b, c, d, e, f, g match the spectrum of fg in the f -table.

Second proof: G_{405} is not a zero-divisor graph since G_{405} is a complete bipartite graph together with ends emanating from two distinct the vertices [5, Theorem IV.5].

Example 5.4. The connected graph G_{475} with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{b, c, d, e\}$, $N(b) = \{a, c, d\}$, $N(c) = \{a, b, d, g\}$, $N(d) = \{a, b, c, f\}$, $N(e) = \{a\}$, $N(f) = \{d\}$, $N(g) = \{c\}$ is not a zero-divisor graph. If we remove f from the graph, we get a zero-divisor graph with six vertices. This is a complete graph on four vertices together with three ends emanating from three different vertices. By [5, Theorem IV.3], we have that G is not a zero-divisor graph.

Example 5.5. G_{482} is not a zero-divisor graph although it satisfies the necessary condition \star . G_{482} can be defined by $N(1) = \{2, 3, 6, 7\}$, $N(2) = \{1, 3\}$, $N(3) = \{1, 2, 4, 6\}$, $N(4) = \{3\}$, $N(5) = \{6\}$, $N(6) = \{1, 3, 5, 7\}$, $N(7) = \{1, 6\}$. It is easy to check that G_{482} satisfies the necessary condition \star .

We have the following,

$$N(1) \cup N(4) = N(1) \cup N(5) = \{2, 3, 6, 7\} = N(1)$$

$$N(3) \cup N(5) = N(3) \cup N(7) = \{1, 2, 4, 6\} = N(3)$$

$$N(6) \cup N(2) = N(6) \cup N(4) = \{1, 3, 5, 7\} = N(6)$$

\bullet	1	2	3	4	5	6	7
1		0	0	1	1	0	0
2	0		0			6	
3	0	0		0	3	0	3
4	1		0			6	
5	1		3			0	
6	0	6	0	6	0		0
7	0		3			0	

Checking the 4- table, we get $4 \bullet 4 = 4$ and $4 \bullet 5 = 1$. Hence $1 \bullet 1 = 1$. It follows that $4 \bullet 7 = 6$ and $6 \bullet 6 = 0$. Hence $2 \bullet 4 = 2$. But then we get $2 \bullet 5 = 0$. This is a contradiction.

Example 5.6. G_{490} is not a zero-divisor graph although it satisfies the necessary condition \star . G_{490} can be defined by $N(1) = \{2\}$, $N(2) = \{1, 3, 5, 7\}$, $N(3) = \{2, 4\}$, $N(4) = \{3, 5, 7\}$, $N(5) = \{2, 4, 6, 7\}$, $N(6) = \{5\}$, $N(7) = \{2, 4, 5\}$. It is easy to check that G_{490} satisfies the necessary condition \star .

We have the following,

$$N(2) \cup N(4) = N(2) \cup N(6) = \{1, 3, 5, 7\} = N(2)$$

$$N(1) \cup N(5) = N(3) \cup N(5) = \{2, 4, 6, 7\} = N(5)$$

$$N(1) \cup N(4) = \{2, 3, 5, 7\} = \overline{N(2)}$$

•	1	2	3	4	5	6	7
1		0		2	5		
2	0	0	0	2	0	2	0
3		0		0	5		
4	2	2	0		0		0
5	5	0	5	0		0	0
6		2			0		
7		0		0	0		

Checking the 4- table, we get $4 \bullet 4 = 4$ and $4 \bullet 6 = 4$. Checking the 6- table, we get $1 \bullet 6 = 2$. It follows that $N(6 \bullet 7) \supseteq \{1, 2, 4, 5\}$. Hence $6 \bullet 7$ is undefined.

Example 5.7. Let $H = G504$ be graph with seven vertices a, b, c, x, y, z, w . Let $N(a) = N(b) = \{x, z\}$, $N(c) = \{x, y\}$, $N(x) = \{a, b, c, z, w\}$, $N(y) = \{c, z\}$, $N(z) = \{a, b, x, y\}$. $N(w) = \{x\}$.

Notice $|N(x) - \{w\}| = 4$. If we remove w from H , we get a graph (2) in Theorem 5.2 which is not a zero-divisor graph with six vertices. Hence by Lemma 5.1, H is not a zero-divisor graph.

Example 5.8. The connected graph $G600$ with $V(G) = \{a, b, c, x, y, z, w\}$ and $E(G)$ defined by $N(a) = \{x, y, z\}$, $N(b) = \{y, z\}$, $N(c) = \{x\}$, $N(x) = \{a, c, y, z\}$, $N(y) = \{a, b, x, z, w\}$, $N(z) = \{a, b, x, y\}$, $N(w) = \{y\}$ is not a zero-divisor graph. Since

$$N(b) \cup N(x) = N(w) \cup N(x) = \{a, c, y, z\} = N(x),$$

and

$$N(c) \cup N(y) = N(y)$$

we have the following multiplication table.

•	a	b	c	x	y	z	w
a				0	0	0	
b				x	0	0	
c				0	y		
x	0	x	0		0	0	x
y	0	0	y	0		0	0
z	0	0		0	0		
w			x	0			

Claim: $cz = z$.

Since

$$N(c) \cup N(z) = \{a, b, x, y\} \subset N(z) \cap \overline{N(y)}$$

we have $cz = y$ or $cz = z$.

Let's check the z - table. If $cz = y$, then $N(zw) \supset \{a, b, c, x, y\}$. It follows that we can not define zw . Hence $cz = z$.

Claim: $cw = z$, $ac = y$, $bc = y$, $y^2 = 0$.

Since

$$N(c) \cup N(w) \subset \overline{N(a)} \cap \overline{N(x)} \cap \overline{N(y)} \cap \overline{N(z)}$$

we have $cw \in \{a, x, y, z\}$.

Let's check the c - table. If $cw = a$, then $N(cw) \supset \{x, y, z\}$. Hence $N(z) \supset \{a, b, x, y, w\}$. Contradiction! If $cw = x$, then $N(cw) \supset \{a, c, y, z\} \cup \{x, y\} = \{a, c, x, y, z\}$. Hence $N(z) \supset \{a, b, x, y, w\}$. Contradiction! If $cw = y$, then $N(cw) \supset \{a, b, x, y, z, w\}$. Hence $N(z) \supset \{a, b, x, y, w\}$. Contradiction! Hence $cw = z$. It follows that $N(ac) \supset \{x, y, z, w\}$. Hence $ac = y$, $y^2 = 0$. Similarly $bc = y$.

Let's Check a - table. Notice $N(ac) \supset \{c, x, y, z\}$ and $N(aw) \supset \{c, x, y, z\}$. It follows that $ab = x$, $aw = x$, and $a^2 = 0$.

Let's check the b - table. Notice $N(b^2) \supset \{c, y, z\}$ and $N(bw) \supset \{c, y, z\}$. Hence $b^2 = x$ and $bw = x$. But $b^2a = x \neq 0 = xa$ and $bwa = x \neq 0 = xa$. It follows that b^2 and bw are undefined.

This argument shows that the above graph is not a zero-divisor graph.

Example 5.9. G_{602} is not a zero-divisor graph.

G_{602} can be defined by $N(1) = \{2\}$, $N(2) = \{1, 3, 5, 7\}$, $N(3) = \{2, 4, 5, 7\}$, $N(4) = \{3\}$, $N(5) = \{2, 3, 6, 7\}$, $N(6) = \{5, 7\}$, $N(7) = \{2, 3, 5, 6\}$. It is easy to check that G_{602} satisfies the \star condition.

We have the following table.

•	1	2	3	4	5	6	7
1		0	3		$\{5, 7\}$		$\{5, 7\}$
2	0		0	2	0	2	0
3	3	0		0	0		0
4		2	0		$\{5, 7\}$		$\{5, 7\}$
5	$\{5, 7\}$	0	0	$\{5, 7\}$		0	0
6		2			0		0
7	$\{5, 7\}$	0	0	$\{5, 7\}$	0	0	

Checking the 4-table, we get $4 \times 6 = 2$ and $2 \times 2 = 2$. Then we have $N(1 \times 4) \supseteq \{2, 3, 6\}$. Hence $1 \times 4 = 5$ or $1 \times 4 = 7$. But $1 \times 4 \neq 5$ since $(1 \times 4) \times 7 \neq 0$. Similarly $1 \neq 7$ since $(1 \times 4) \times 5 \neq 0$.

Example 5.10. G_{607} is not a zero-divisor graph.

G_{607} can be defined by $N(1) = \{2\}$, $N(2) = \{1, 3, 5, 6\}$, $N(3) = \{2, 4, 6\}$, $N(4) = \{3, 5, 6\}$, $N(5) = \{2, 4, 6, 7\}$, $N(6) = \{2, 3, 4, 5\}$, $N(7) = \{5\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0			5	6	6
2	0		0	2	0	0	2
3		0		0	5	0	
4		2	0		0	0	
5	5	0	5	0		0	0
6	6	0	0	0	0		6
7	6	2			0	6	

Checking 1–table, we get $6 \times 6 = 6$. It follows that $1 \times 4 = 5$ and $5 \times 5 = 0$. But then we can't define 1×3 .

Example 5.11. Let $H = G617$ be graph with seven vertices a, b, c, x, y, z, w . Let $N(a) = \{x, y\}$, $N(b) = \{x, z\}$, $N(c) = \{y, z\}$, $N(x) = \{a, c, y, z, w\}$, $N(y) = \{a, b, x, z\}$, $N(z) = \{a, c, x, y\}$. $N(w) = \{x\}$.

Notice $|N(x) - \{w\}| = 4$. If we remove w from H , we get a graph (3) in Theorem 5.2 that is not a zero-divisor graph which is proved in [2]. Hence by Lemma 5.1, H is not a zero-divisor graph.

Example 5.12. $G627$ is not a zero-divisor graph. $G627$ can be defined by $N(1) = \{2\}$, $N(2) = \{1, 3, 5, 7\}$, $N(3) = \{2, 4, 5\}$, $N(4) = \{3, 5\}$, $N(5) = \{2, 3, 4, 6, 7\}$, $N(6) = \{5, 7\}$, $N(7) = \{2, 5, 6\}$. It is easy to check that $G627$ satisfies the \star condition.

We have the following table.

•	1	2	3	4	5	6	7
1		0	$\{3, 5\}$	$\{2, 3, 5\}$	5	$\{2, 5, 7\}$	$\{5, 7\}$
2	0		0	2	0	2	0
3	$\{3, 5\}$	0		0	0	5	5
4	$\{2, 3, 5\}$	2	0		0	$\{2, 5\}$	5
5	5	0	0	0	0	0	0
6	$\{2, 5, 7\}$	2	5	$\{2, 5\}$	0		0
7	$\{5, 7\}$	0	5	5	0	0	

Checking 3–table, we get $1 \times 3 = 3$. Checking 6–table, we get $4 \times 6 = 2$, $6 \times 6 = 2$, and $2 \times 2 = 2$. But then we can't define 1×6 .

Example 5.13. The connected graph $G635$ with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{b, d, g\}$, $N(b) = \{a, c, e, g\}$, $N(c) = \{b, g\}$, $N(d) = \{a, e\}$, $N(e) = \{b, d, f, g\}$, $N(f) = \{e\}$, $N(g) = \{a, b, c, e\}$ is not a zero-divisor graph. If we remove g , we get graph (1) in Theorem 5.2.

Notice

$$N(a) \cup N(e) = N(c) \cup N(e) = \{b, d, f, g\} = N(e)$$

and

$$N(a) \cup N(f) = \{b, d, e, g\} \subset \overline{N(e)}$$

we have the following multiplication table.

•	a	b	c	d	e	f	g
a		0		0	e	e	0
b	0		0		0		0
c		0			e		0
d	0				0		
e	e	0	e	0	0	0	0
f	e				0		
g	0	0	0		0		

By checking a -table, we get $ac = a^2 = a$. It follows that by checking c -table, we get $cf = e$. Since $N(c) \cup N(d) = \{a, b, e, g\}$ is a subset of $\overline{N(b)}$ or a subset of $\overline{N(g)}$, we have $cd = b$ or $cd = g$.

If $cd = g$, then $fg = f(cd) = (fc)d = ed = 0$. This is a contradiction.

If $cd = b$, then $fb = f(cd) = (fc)d = ed = 0$. Contradiction.

Example 5.14. The connected graph $G669$ with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{b, c, d, e, f\}$, $N(b) = \{a, d, e, f, g\}$, $N(c) = \{a, g\}$, $N(d) = \{a, b\}$, $N(e) = \{a, b\}$, $N(f) = \{a, b\}$, $N(g) = \{b, c\}$ is not a zero-divisor graph. If we remove f , we get graph (2) in Theorem 5.2.

We have the following multiplication table.

•	a	b	c	d	e	f	g
a		0	0	0	0	0	
b	0			0	0	0	0
c	0						0
d	0	0					
e	0	0					
f	0	0					
g		0	0				

Since $N(c) \cup N(d) = \{a, b, g\} \subset \overline{N(b)}$. Hence $b^2 = 0$. Since $N(d) \cup N(g) = \{a, b, c\} \subset \overline{N(a)}$. Hence $a^2 = 0$ and we have a updated multiplication Table.

•	a	b	c	d	e	f	g
a	0	0	0	0	0	0	
b	0	0		0	0	0	0
c	0						0
d	0	0					
e	0	0					
f	0	0					
g		0	0				

Since $N(b) \cup N(c) = \{a, d, e, f, g\}$, we get $bc = b$. We get the following updated multiplication Table.

•	a	b	c	d	e	f	g
a	0	0	0	0	0	0	
b	0	0	b	0	0	0	0
c	0	b					0
d	0	0					
e	0	0					
f	0	0					
g		0	0				

Since $N(a) \cup N(g) = \{b, c, d, e, f\}$, we have $ag = a$. We have the following updated Multiplication Table.

•	a	b	c	d	e	f	g
a	0	0	0	0	0	0	a
b	0	0	b	0	0	0	0
c	0	b					0
d	0	0					
e	0	0					
f	0	0					
g	a	0	0				

Since

$$N(c) \cup N(d) = N(c) \cup N(e) = N(c) \cup N(f) = \{a, b, g\} \subset \overline{N(b)}$$

we have $cd = ce = cf = b$ and the following multiplication Table.

•	a	b	c	d	e	f	g
a	0	0	0	0	0	0	a
b	0	0	b	0	0	0	0
c	0	b		b	b	b	0
d	0	0	b				
e	0	0	b				
f	0	0	b				
g	a	0	0				

Since

$$N(g) \cup N(d) = N(g) \cup N(e) = N(g) \cup N(f) = \{a, b, c\} \subset \overline{N(a)}$$

Hence $gd = eg = gf = a$. We have the following multiplication Table.

•	a	b	c	d	e	f	g
a	0	0	0	0	0	0	a
b	0	0	b	0	0	0	0
c	0	b		b	b	b	0
d	0	0	b				a
e	0	0	b				a
f	0	0	b				a
g	a	0	0	a	a	a	

By checking g - table, we get $g^2 = g$. By checking c - table, we get $c^2 = c$. By checking d - table, we showed that there is no way to define de or df .

Therefore G is not a zero-divisor graph.

Example 5.15. The connected graph $G677$ with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{b, d\}$, $N(b) = \{a, c, e, f, g\}$, $N(c) = N(f) = \{b, e\}$, $N(d) = \{a, e, g\}$, $N(e) = \{b, c, d, f\}$, $N(g) = \{b, d\}$ is not a zero-divisor graph. If we remove g , we get graph (2) in Theorem 5.2.

Notice

$$\begin{aligned} N(a) \cup N(c) &= N(a) \cup N(f) = \{b, d, e\} \subset \overline{N(e)} \\ N(g) \cup N(c) &= N(g) \cup N(f) = \{b, d, e\} \subset \overline{N(e)} \\ N(a) \cup N(e) &= N(g) \cup N(e) = \{b, c, d, f\} = N(e) \\ N(b) \cup N(d) &= \{a, c, e, f, g\} = N(b) \end{aligned}$$

we get the following multiplication table.

•	a	b	c	d	e	f	g
a		0	e	0	e	e	0
b	0		0	b	0	0	0
c	e	0			0		e
d	0	b			0		0
e	e	0	0	0	0	0	e
f	e	0			0		e
g		0	e	0	e	e	

By checking c -table, we get $cd = cf = b$. It follows that $b = bd = fcd = fb = 0$. This is a contradiction.

Example 5.16. The connected graph $G742$ with $V(G) = \{a, b, c, x, y, z, w\}$ and $E(G)$ defined by $N(a) = \{c, x, y, z, w\}$, $N(b) = \{x\}$, $N(c) = \{a, x, y, z\}$, $N(x) = \{a, b, c, z\}$, $N(y) = \{a, c, z\}$, $N(z) = \{a, c, x, y\}$, $N(w) = \{a\}$ is not a zero-divisor graph. Since

$$N(a) \cup N(b) = \{c, x, y, z, w\} = N(a)$$

and

$$N(x) \cup N(y) = N(x) \cup N(w) = \{a, b, c, z\} = N(x)$$

one gets $ab = a$, $xy = xw = x$ and the following multiplication table.

•	a	b	c	x	y	z	w
a		a	0	0	0	0	a
b	a			0			
c	0			0	0	0	
x	0	0	0		x	0	x
y	0		0	x		0	
z	0		0	0	0		
w	0			x			

Since

$$N(b) \cup N(y) = \{a, c, x, z\},$$

one gets that $by \in \{a, c, z, x\}$. By checking y -table, one gets $y^2 \in \{x, y\}$.

If $y^2 = y$, by checking y -table, one gets $by = x$ and $yw = y$. By checking b -table, one gets that $bz = a$ and $bw = x$. Then one can't define b^2 .

Suppose $y^2 = x$. By checking y -table, we get $x^2 = x$. It follows that $N(by) \supset \overline{N(a)}$ or $N(by) \supset \overline{N(c)}$. Hence one has $by \in \{a, c\}$ and $yw \in \{x, y\}$. If $by = a$ and $yw = y$,

then $(wy)b = a \neq 0 = w(yb)$. If $by = c$ and $yw = x$, one gets $cw = byw = bx = 0$. It follows that $by = a$, $yw = x$ or $by = c$, $yw = y$. If $by = c$, then $cw = c$.

Case I: $by = a$ and $yw = x$.

Suppose $by = a$ and $yw = x$. By checking b - table, one gets $b^2 = b$ and $N(bw) \supset \{a, x, y\}$. It follows that $bw \in \{a, c, z\}$. Suppose $by = a$, $yw = x$, and $bw = a$. By checking b - table, one gets $w^2 = w$ and $N(cw) \supset \{a, b, x, y, w\}$. Hence one can't define cw . Suppose $by = a$, $yw = x$, and $bw = c$. By checking b - table, one gets $N(wz) \supset \{a, b, x, y\}$. Hence one can't define wz . Suppose $by = a$, $yw = x$, and $bw = z$. By checking b - table, one gets $N(cw) \supset \{a, b, x, y\}$. Hence one can't define cw .

Case II: $by = c$ and $yw = y$.

We get $cw = c$ and $w^2 = w$. By checking w - table, one gets $N(wz) \supset \{a, c, x, y\}$. Hence $wz \in \{a, c, z\}$. If $wz = a$, then $(zw)w = 0 \neq z(w^2)$. If $wz = c$, then $z^2 = 0$. By checking w - table and the spectrum of each vertex of G , one can't define bw . Suppose $wz = z$. By checking w - table and the spectrum of each vertex of G , then one can't define bw .

Example 5.17. Let $H = G750$ be graph with seven vertices a, b, c, x, y, z, w . Let $N(a) = \{b, x, y, z\}$, $N(b) = \{a, y\}$, $N(c) = \{x, z\}$, $N(x) = \{a, c, y, z, w\}$, $N(y) = \{a, b, x, z\}$, $N(z) = \{a, c, x, y\}$. $N(w) = \{x\}$.

Notice $|N(x) - \{w\}| = 4$. If we remove w from H , we get a graph (4) in Theorem 5.2 that is not a zero-divisor graph which is proved in [2]. Hence by Lemma 5.1, H is not a zero-divisor graph.

Example 5.18. $G754$ is not a zero-divisor graph.

$G754$ can be defined by $N(1) = \{2\}$, $N(2) = \{1, 3, 5, 6, 7\}$, $N(3) = \{2, 4, 6\}$, $N(4) = \{3, 5, 6\}$, $N(5) = \{2, 4, 6\}$, $N(6) = \{2, 3, 4, 5, 7\}$, $N(7) = \{2, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0		$\{2, 6\}$		6	$\{2, 3, 5, 6, 7\}$
2	0		0	2	0	0	0
3		0		0		0	
4	$\{2, 6\}$	2	0		0	0	$\{2, 6\}$
5		0		0		0	
6	6	0	0	0	0		0
7	$\{2, 3, 5, 6, 7\}$	0		$\{2, 6\}$		0	

Let $1 \times 4 = 2$. Then $2 \times 2 = 0$. Checking 1- table, we get $1 \times 1 = 6$. Then we can't define 1×7 .

Let $1 \times 4 = 6$. Then $6 \times 6 = 0$. Checking 4- table, we get $4 \times 7 = 2$. Then we can't define 4×4 .

Example 5.19. The connected graph $G766$ with $V(G) = \{a, b, c, x, y, z, w\}$ and $E(G)$ defined by $N(a) = \{c, x, y, z\}$, $N(b) = \{x, y, z\}$, $N(c) = \{a, y\}$, $N(x) = \{a, b, y, z, w\}$, $N(y) = \{a, b, c, x\}$, $N(z) = \{a, b, x\}$, $N(w) = \{x\}$ is not a zero-divisor graph. If we remove w from the graph, we get the graph (1) in Theorem 5.1 which is a zero-divisor graph with six vertices. Since

$$N(a) \cup N(b) = N(a) \cup N(w) = \{c, x, y, z\} = N(a)$$

$$N(y) \cup N(z) = N(y) \cup N(w) = \{a, b, c, x\} = N(y)$$

$$N(x) \cup N(c) = \{a, b, y, z, w\} = N(x)$$

we get $ab = aw = a$, $yz = yw = y$ and $xc = x$. we have the following multiplication table.

•	a	b	c	x	y	z	w
a		a	0	0	0	0	a
b	a			0	0	0	
c	0			x	0	y	
x	0	0	x		0	0	0
y	0	0	0	0		y	y
z	0	0		0	y		
w	a			0	y		

Since

$$N(b) \cup N(c) = \{a, x, y, z\}$$

and

$$N(c) \cup N(z) = \{a, b, x, y\}$$

we get

$$bc = \begin{cases} a & \text{if } a^2 = 0 \\ x & \text{if } x^2 = 0 \end{cases}$$

and

$$cz = \begin{cases} x & \text{if } x^2 = 0 \\ y & \text{if } y^2 = 0 \end{cases}$$

Case I: $cz = x$ and $bc = a$. By checking c -table, we can't define c^2 .

Case II: $cz = x$ and $bc = x$. By checking c -table, we get $cw = x$. Then checking w -table, we can't define w^2 .

Case III: $cz = y$ and $bc = a$. By checking c -table, we get $c^2 = x$. It follows that we can't define cw .

Case IV: $cz = y$ and $bc = x$. By checking c -table, we can't define c^2 .

This argument shows that the graph is not a zero-divisor graph.

Example 5.20. G_{772} is not a zero-divisor graph. G_{772} can be defined by $N(1) = \{2, 4\}$, $N(2) = \{1, 3, 5, 6\}$, $N(3) = \{2, 4, 5\}$, $N(4) = \{1, 3, 5, 6\}$, $N(5) = \{2, 3, 4, 6\}$, $N(6) = \{2, 4, 5, 7\}$, $N(7) = \{6\}$.

We have the following table.

•	1	2	3	4	5	6	7
1		0		0	5	6	$\{5, 6\}$
2	0		0	$\{2, 4\}$	0	0	$\{2, 4\}$
3		0		0	0	6	$\{5, 6\}$
4	0	$\{2, 4\}$	0		0	0	$\{2, 4\}$
5		0	0	0		0	5
6	6	0	6	0	0		0
7	$\{5, 6\}$	$\{2, 4\}$	$\{5, 6\}$	$\{2, 4\}$	5	0	

Checking 7-table, we get $7 \times 7 = 7$. Let $3 \times 7 = 5$. Then we can't define 1×7 . Let $3 \times 7 = 6$. Then we get $3 \times 7 = 0$. This is a contradiction.

Example 5.21. The connected graph $G793$ which is defined by $N(a) = \{b, f\}$, $N(b) = \{a, c, d, f, g\}$, $N(c) = \{b, d\}$, $N(d) = \{b, c, e, f\}$, $N(e) = \{d, f\}$, $N(f) = \{a, b, d, e, g\}$, $N(g) = \{b, f\}$ is not a zero-divisor graph. If we remove g , we get graph (3) in Theorem 5.2. Notice

$$N(a) \cup N(d) = N(g) \cup N(d) = \{b, c, e, f\} = N(d)$$

$$N(b) \cup N(e) = \{a, c, d, f, g\} = N(b)$$

$$N(c) \cup N(f) = \{a, b, d, e, g\} = N(f)$$

We have the following multiplication table.

•	a	b	c	d	e	f	g
a		0		d		0	
b	0		0	0	b	0	0
c		0		0		f	
d	d	0	0		0	0	d
e		b		0		0	
f	0	0	f	0	0		0
g		0		d		0	

Notice

$$N(a) \cup N(c) = N(a) \cup N(e) = \{b, d, f\} = N(c) \cup N(e) \subset \overline{N(b)} \cap \overline{N(d)} \cap \overline{N(f)}$$

Hence $ac \in \{b, d, f\}$, $ae \in \{b, d, f\}$ and $ce \in \{b, d, f\}$.

If $ac = b$, then $b^2 = 0$. Checking a -table, we get that if $ac = b$, then $ce \notin \{b, d, f\}$.

If $ac = d$, then $d^2 = 0$. Checking c -table, we get $ce \notin \{b, d\}$. If $ce = f$, then $f^2 = 0$ and $N(c^2) \supset \{a, b, d\}$. It follows that $c^2 = b$ or $c^2 = f$. Checking the spectrum of b and f , we have $c^2 \neq b$ or $c^2 \neq f$.

If $ac = f$ and $ce = b$, then by checking c -table, we get $N(c^2) \supset \{b, d, e\}$. Hence $c^2 \in \{d, f\}$. Checking the spectrum of d and f , we get $c^2 \notin \{d, f\}$.

If $ac = f$, then by checking c -table, we get $ce \neq d$.

If $ce = f$, then by checking e -table, we get $N(ae) \supset \{b, c, d, f\}$. Hence $ae \in \{b, d\}$. If $ae = b$ or $ae = d$, by checking e -table, we have that e^2 is undefined.

Example 5.22. $G799$ is not a zero-divisor graph. $G799$ can be defined by $N(1) = \{2, 3, 6\}$, $N(2) = \{1, 3, 6\}$, $N(3) = \{1, 2, 4, 6, 7\}$, $N(4) = \{3, 5\}$, $N(5) = \{4, 6\}$, $N(6) = \{1, 2, 3, 5, 7\}$, $N(7) = \{3, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0	6	3	0	$\{1, 2, 3, 6\}$
2	0		0	6	3	0	
3	0	0	0	0	3	0	0
4	6	6	0		0	6	6
5	3	3	3	0		0	3
6	0	0	0	6	0	0	0
7	$\{1, 2, 3, 6\}$		0	6	3	0	

Checking 1-table, we can't define 1×7 .

Example 5.23. G_{803} is not a zero-divisor graph. G_{803} can be defined by $N(1) = \{2, 3\}$, $N(2) = \{1, 3, 5, 6, 7\}$, $N(3) = \{1, 2, 4, 6\}$, $N(4) = \{3, 5\}$, $N(5) = \{2, 4, 6\}$, $N(6) = \{2, 3, 5, 7\}$, $N(7) = \{2, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0		3		
2	0		0	2	0	0	0
3	0	0	0	0	3	0	3
4		4	0		0		
5	3	0	3	0		0	
6		0	0		0		0
7		0	3			0	

Checking 5-table, we get $5 \times 5 = 5$ and $5 \times 7 = 5$. Checking 7-table, we get $1 \times 7 = 3$. It follows $4 \times 7 = 2$ and $2 \times 2 = 0$. But then $0 = 2 \times 2 = 2 \times 4 \times 7 = 4 \times 7 = 2$. This is a contradiction.

Example 5.24. G_{808} is not a zero-divisor graph. G_{808} can be defined by $N(1) = \{2, 4, 5, 6, 7\}$, $N(2) = \{1, 3\}$, $N(3) = \{2, 5, 6\}$, $N(4) = \{1, 5\}$, $N(5) = \{1, 3, 4, 6\}$, $N(6) = \{1, 3, 5, 7\}$, $N(7) = \{1, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	1	0	0	0	0
2	0		0		5	6	
3	3	0		1	0	0	1
4	0		1		0	6	
5	0	5	0	0		0	5
6	0	6	0	6	0		0
7	0		1		5	0	

Checking 2-table, we get $2 \times 2 = 2$. It follows that $2 \times 4 = 6$ and $2 \times 7 = 5$. Checking 7-table, we get $N(4 \times 7) \supseteq \{1, 2, 3, 5, 6\}$. Hence we can't define 4×7 .

Example 5.25. G_{893} is not a zero-divisor graph. G_{893} can be defined by $N(1) = \{2\}$, $N(2) = \{1, 3, 4, 6, 7\}$, $N(3) = \{2, 4, 6, 7\}$, $N(4) = \{2, 3, 5, 7\}$, $N(5) = \{4, 6\}$, $N(6) = \{2, 3, 5, 7\}$, $N(7) = \{2, 3, 4, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0		$\{4, 6\}$	$\{2, 3, 7\}$	$\{4, 6\}$	
2	0		0	0	2	0	0
3		0		0		0	0
4	$\{4, 6\}$	0	0		0	$\{4, 6\}$	0
5	$\{2, 3, 7\}$	2		0		0	
6	$\{4, 6\}$	0	0	$\{4, 6\}$	0		0
7		0	0	0		0	

Checking 1-table. If $1 \times 5 = 2$ or $1 \times 5 = 3$, then $N(1 \times 7) \supseteq \{2, 3, 4, 5, 6\}$. It follows that we can't define 1×7 . Hence $1 \times 5 = 7$. But then $N(1 \times 3) \supseteq \{2, 4, 5, 6, 7\}$. Hence we can't define 1×3 .

Example 5.26. G_{899} is not a zero-divisor graph. G_{899} can be defined by $N(1) = \{2, 3, 5\}$, $N(2) = \{1, 3, 6\}$, $N(3) = \{1, 2, 4, 5, 6\}$, $N(4) = \{3, 5, 6\}$, $N(5) = \{1, 3, 4, 6\}$, $N(6) = \{2, 3, 4, 5, 7\}$, $N(7) = \{6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0		0	6	
2	0		0			0	
3	0	0		0	0	0	3
4			0		0	0	
5	0		0	0		0	
6	6	0	0	0	0		0
7			3			0	

We know that $1 \times 7 = 3$ or $1 \times 7 = 6$.

We have $1 \times 4 = 3$ or $1 \times 4 = 6$. If $1 \times 4 = 3$, then $3 \times 3 = 0$. But then we get $0 = (1 \times 7) \times 4 = 3$. This is a contradiction. Hence we can't define 1×7 .

If $1 \times 4 = 6$, then $6 \times 6 = 0$. Hence $1 \times 1 = 1$. Let $1 \times 7 = 3$. Then it follows that $1 \times 7 = 0$. This is a contradiction. Hence $1 \times 7 = 6$.

Checking 7-table, we get $7 \times 7 = 3$. But then $N(4 \times 7) \supseteq \{1, 3, 6, 7\}$. Hence we can't define 4×7 .

Example 5.27. The connected graph G_{907} with $V(G) = \{a, b, c, x, y, z, w\}$ and $E(G)$ defined by $N(a) = \{c, x, y, z, w\}$, $N(b) = \{c, x, y, z\}$, $N(c) = \{a, b, x, y\}$, $N(x) = \{a, b, c, z\}$, $N(y) = \{a, b, c\}$, $N(z) = \{a, b, x\}$, $N(w) = \{a\}$ is not a zero-divisor graph.

$$N(a) \cup N(b) = \{c, x, y, z, w\}$$

$$N(b) \cup N(w) = \{a, c, x, y, z\}$$

$$N(c) \cup N(z) = N(c) \cup N(w) = \{a, b, x, y\}$$

$$N(x) \cup N(y) = N(c) \cup N(w) = \{a, b, c, z\}$$

we have the following multiplication table.

•	a	b	c	x	y	z	w
a	0	a	0	0	0	0	0
b	a		0	0	0	0	a
c	0	0		0	0	c	c
x	0	0	0		x	0	x
y	0	0	0	x			
z	0	0	c	0			
w	0	a	c	x			

Checking the w -table, we get that w^2 is undefined.

Example 5.28. G_{917} is not a zero-divisor graph. G_{917} can be defined by $N(1) = \{2, 7\}$, $N(2) = \{1, 3, 5, 7\}$, $N(3) = \{2, 4, 5, 7\}$, $N(4) = \{3, 5\}$, $N(5) = \{2, 3, 4, 6, 7\}$, $N(6) = \{5, 7\}$, $N(7) = \{1, 2, 3, 5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0			5		0
2	0		0	{2,7}	0	{2,7}	0
3		0		0	0		0
4		{2,7}	0		0		4
5	5	0	0	0		0	0
6		{2,7}			0		0
7	0	0	0	4	0	0	

Suppose $2 \times 4 = 7$. Checking 2-table, we get $N(2 \times 6) \supseteq \{1, 3, 4, 5, 7\}$. Hence we can't define 2×6 . It follows that $2 \times 4 = 2$. Checking 4-table, we have $0 = 2 \times 7 = 2 \times 4 \times 7 = 2 \times 4 = 2$. This is a contradiction.

Example 5.29. The connected graph G_{918} which is defined by $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f, g\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c, e\}$, $N(e) = \{b, c, d, f\}$, $N(f) = \{a, b, c, e, g\}$, $N(g) = \{b, f\}$ is not a zero-divisor graph. If we remove g , we get graph (4) in Theorem 5.2.

We claim that if $a^2 = g$, then $g^2 \neq a$.

Suppose $a^2 = g$ and $g^2 = a$. Then $(ag)^2 = agag = a^2g^2 = ga = ag$. It follows that $a(ag) = g^2 = a$.

Notice $N(a) \cup N(g) = \{b, f\} \subset N(a) \cap N(g) \cap \overline{N(b)} \cap N(c) \cap N(e) \cap \overline{N(f)}$.

If $ag = a$, then $a(ag) = a^2 = g$. This is a contradiction. If $ag = g$, then $ag = a(ag) = a$. Another contradiction. Hence $ag \neq a$ or $ag \neq g$.

If $ag = b$, then $a = a(ag) = ab = 0$. This is a contradiction.

Since $N(a) \cup N(c) = \{b, d, e, f\} \subset N(c) \cup \overline{N(e)}$, $ac \in \{c, e\}$. It follows that $ac \neq a$. Hence $ag \neq c$.

Since $N(a) \cup N(e) = \{b, c, d, f\} \subset \overline{N(c)} \cap N(e)$, $ae \in \{c, e\}$. It follows that $ae \neq a$. Hence $ag \neq e$.

If $ag = f$, then $a = a(ag) = af = 0$. Contradiction.

It is easy to check that $N(x) \cup N(y)$ is not a subset of $\overline{N(a)}$ or $\overline{N(g)}$ unless $x = a$, $y = g$ or $x = g$, $y = a$.

Suppose G is a zero-divisor graph, then by the above argument, we get the graph formed by removing a or g from G is a zero-divisor graph. But this is a contradiction to the fact showed in [2].

Example 5.30. G_{928} is not a zero-divisor graph. G_{928} can be defined by $N(1) = \{2, 5\}$, $N(2) = \{1, 3, 5, 6, 7\}$, $N(3) = \{2, 4, 6\}$, $N(4) = \{3, 5\}$, $N(5) = \{1, 2, 4, 6, 7\}$, $N(6) = \{2, 3, 5, 7\}$, $N(7) = \{2, 5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	5		0		
2	0		0	2	0	0	0
3	5	0	3	0	5	0	5
4		2	0		0		
5	0	0	5	0	0	0	0
6		0	0		0		0
7		0	5		0	0	

We know $1 \times 6 = 2$ or $1 \times 6 = 6$. Suppose $1 \times 6 = 2$. Checking 6-table, we get $4 \times 6 = 6$. Then checking 4-table, we get $4 \times 4 = 4$. It follows that we can't define 1×4 .

Suppose $1 \times 6 = 6$. Checking 1-table, we get $1 \times 1 = 6$. It follows that $1 \times 7 = 2$. But then we can't define 1×4 .

Example 5.31. $G933$ is not a zero-divisor graph. $G933$ can be defined by $N(1) = \{2, 5, 6, 7\}$, $N(2) = \{1, 3, 5, 6, 7\}$, $N(3) = \{2, 4\}$, $N(4) = \{3, 5\}$, $N(5) = \{1, 2, 4, 6, 7\}$, $N(6) = \{1, 2, 5\}$, $N(7) = \{1, 2, 5\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	5	2	0	0	0
2	0	0	0	2	0	0	0
3	5	0		0	5	5	5
4	2	2	0		0	2	2
5	0	0	5	0	0	0	0
6	0	0	5	2	0		
7	0	0	5	2	0		

Checking 6-table, we get $N(6 \times 7) \supseteq \{1, 2, 3, 4, 5\}$. Hence we can't define 6×7 .

Example 5.32. $G938$ is not a zero-divisor graph. $G938$ can be defined by $N(1) = \{2, 6, 7\}$, $N(2) = \{1, 3, 5, 6\}$, $N(3) = \{2, 4, 6, 7\}$, $N(4) = \{3, 7\}$, $N(5) = \{2, 7\}$, $N(6) = \{1, 2, 3, 7\}$, $N(7) = \{1, 3, 4, 5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	3			0	0
2	0		0	7	0	0	7
3	3	0		0	3	0	0
4		7	0			6	0
5		0	3			6	0
6	0	0	0	6	6		0
7	0	7	0	0	0	0	0

Checking 4-table, we get $4 \times 4 = 4 \times 5 = 6$. Then we can't define 1×4 because $N(1 \times 4) \supseteq \{2, 3, 4, 5, 6, 7\}$.

Example 5.33. $G953$ is not a zero-divisor graph. $G953$ can be defined by $N(1) = \{2, 5\}$, $N(2) = \{1, 3, 7\}$, $N(3) = \{2, 4, 5, 6, 7\}$, $N(4) = \{3, 5, 6\}$, $N(5) = \{1, 3, 4, 6, 7\}$, $N(6) = \{3, 4, 5\}$, $N(7) = \{2, 3, 5\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	3	3	0	3	
2	0		0	5	5	5	0
3	3	0	0	0	0	0	0
4	3	5	0		0	0	3
5	0	5	0	0	0	0	0
6	3	5	0	0	0		3
7		0	0	3	0	3	

Checking 1-table, we get $1 \times 1 = 1$ and $1 \times 7 = 3$. But then $0 = 3 \times 4 = 1 \times 7 \times 4 = 1 \times 3 = 3$. This is a contradiction.

Example 5.34. The connected graph G_{1024} with $V(G) = \{a, b, c, x, y, z, w\}$ and $E(G)$ defined by $N(a) = N(b) = \{c, x, y, z\}$, $N(c) = N(z) = \{a, b, x, y\}$, $N(x) = \{a, b, c, z, w\}$, $N(y) = \{a, b, c, z\}$, $N(w) = \{x\}$ is not a zero-divisor graph. If we remove w from the graph, we get the graph (3) in Theorem 5.1 which is a zero-divisor graph with six vertices.

Since

$$N(x) \cup N(y) = \{a, b, c, z, w\}$$

and

$$N(y) \cup N(w) = \{a, b, c, x, z\}$$

one gets that $xy = x$, $yw = x$ and $x^2 = 0$. Hence we have the following multiplication table.

•	a	b	c	x	y	z	w
a			0	0	0	0	
b			0	0	0	0	
c	0	0		0	0		
x	0	0	0	0	x	0	0
y	0	0	0	x		0	x
z	0	0		0	0		
w				0	x		

Since

$$N(a) \cup N(w) = N(b) \cup N(w) = \{c, x, y, z\}$$

and

$$N(c) \cup N(w) = N(z) \cup N(w) = \{a, b, x, y\}$$

one gets $aw \in \{a, b\}$, $bw \in \{a, b\}$, $cw \in \{c, z\}$, and $zw \in \{c, z\}$.

By checking w -table and the spectrum of each vertex in G , it follows that w^2 is undefined.

Example 5.35. G_{1030} is not a zero-divisor graph. G_{1030} can be defined by $N(1) = \{2, 5, 6\}$, $N(2) = \{1, 3, 4, 5, 6\}$, $N(3) = \{2, 4, 5, 6\}$, $N(4) = \{2, 3\}$, $N(5) = \{1, 2, 3, 6, 7\}$, $N(6) = \{1, 2, 3, 5, 7\}$, and $N(7) = \{5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0			0	0	
2	0		0	0	0	0	2
3			0	0	0	0	
4		0	0				
5	0	0	0			0	0
6	0	0	0		0		0
7		2			0	0	

We know that $3 \times 7 = 2$ or $3 \times 7 = 3$. If $3 \times 7 = 2$, then $N(1 \times 3) \supseteq \{2, 4, 5, 6, 7\}$. Hence we can't define 1×3 . It follows that $3 \times 7 = 3$.

Checking 4-table, we get $7 \times 7 = 7$ and $4 \times 7 = 2$ or $4 \times 7 = 3$. We know that $4 \times 5 \in \{5, 6\}$ and $4 \times 6 \in \{5, 6\}$. Checking 4-table, we get $N(4 \times 4) \supseteq \{2, 3, 7\}$. Hence $4 \times 4 \in \{0, 5, 6\}$. It follows that we can't define 4×4 .

Example 5.36. G_{1034} is not a zero divisor graph. G_{1034} can be defined by $N(1) = \{2, 3, 4, 5\}$, $N(2) = \{1, 3, 4, 5\}$, $N(3) = \{1, 2, 4, 6\}$, $N(4) = \{1, 2, 3, 5, 7\}$, $N(5) = \{1, 2, 4, 6, 7\}$, $N(6) = \{3, 5\}$, and $N(7) = \{4, 5\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0	0	0		
2	0		0	0	0		
3	0	0		0	5	0	5
4	0	0	0		0	4	0
5	0	0	5	0	0	0	0
6			0	4	0		
7			5	0	0		

We know that $6 \times 7 \in \{1, 2, 4\}$. Let's Check 7-table. If $6 \times 7 = 1$ or $6 \times 7 = 4$, then we can't define 2×7 since $N(2 \times 7) \supseteq \{3, 4, 5, 6\}$. If $6 \times 7 = 2$, then $N(1 \times 7) \supseteq \{3, 4, 5, 6\}$. Hence we can't define 1×7 .

Example 5.37. G_{1043} is not a zero-divisor graph. G_{1043} can be defined by $N(1) = \{2, 3\}$, $N(2) = \{1, 3, 4, 5, 6\}$, $N(3) = \{1, 2, 4, 5, 7\}$, $N(4) = \{2, 3, 5\}$, $N(5) = \{2, 3, 4, 6, 7\}$, $N(6) = \{2, 5, 7\}$, and $N(7) = \{3, 5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0		5		
2	0		0	0	0	0	2
3	0	0		0	0	3	0
4		0	0		0		
5	5	0	0	0		0	0
6		0	3		0		0
7		2	0		0	0	

We know $1 \times 7 \in \{2, 5\}$. Suppose $1 \times 7 = 2$. Checking 1-table, we get $1 \times 1 = 5$ and $1 \times 4 = 3$. Then we can't define 1×6 since $N(1 \times 6) \supseteq \{1, 2, 3, 5, 7\}$ and $(1 \times 6) \times 4 = 3$. Suppose $1 \times 7 = 5$. Checking 7-table, we get $7 \times 7 = 2$. Then we can't define 4×7 since $N(4 \times 7) \supseteq \{1, 2, 3, 5, 6, 7\}$.

Example 5.38. The connected graph G_{1044} with $V(G) = \{a, b, c, d, e, f, g\}$ and $E(G)$ defined by $N(a) = \{b, d, g\}$, $N(b) = \{a, c, e, f, g\}$, $N(c) = N(f) = \{b, e, g\}$, $N(d) = \{a, e\}$, $N(e) = \{b, c, d, f, g\}$, $N(g) = \{a, b, c, e, f\}$ is not a zero-divisor graph. If we remove g , we get graph (2) in Theorem 5.2.

Notice

$$N(a) \cup N(c) = N(a) \cup N(f) = \{b, d, e, g\} \subset \overline{N(e)}$$

and

$$N(a) \cup N(e) = \{b, c, d, f, g\} = N(e)$$

we have the following multiplication table.

•	a	b	c	d	e	f	g
a		0	e	0	e	e	0
b	0		0		0	0	0
c	e	0			0		0
d	0				0		
e	e	0	0	0	0	0	0
f	e	0			0		0
g	0	0	0		0	0	

Notice $N(c) \cup N(d) = \{a, b, e, g\}$ is a subset of $\overline{N(b)}$ or a subset of $\overline{N(g)}$. Hence $cd = b$ or $cd = g$.

If $cd = b$, then $dcf = bf = 0$. If $cd = g$, then $dcf = gf = 0$. In either cases, we have $d \in N(cf)$. By checking c -table, we have $N(cf) \supset \{a, b, d, e, g\}$. Hence cf is undefined.

Example 5.39. G_{1060} is not a zero-divisor graph. G_{1060} can be defined by $N(1) = \{2, 3, 4, 5\}$, $N(2) = \{1, 3, 6\}$, $N(3) = \{1, 2, 4, 6, 7\}$, $N(4) = \{1, 3, 5, 6, 7\}$, $N(5) = \{1, 4, 6\}$, $N(6) = \{2, 3, 4, 5\}$, $N(7) = \{3, 4\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0	0	0		
2	0		0	4		0	
3	0	0		0	3	0	0
4	0	4	0		0	0	0
5	0		3	0		0	
6		0	0	0	0		
7			0	0			

We know that $2 \times 5 \in \{3, 4\}$. Suppose $2 \times 5 = 3$. Checking 7-table, we get $2 \times 7 = 4$. Then we can't define 2×2 since $N(2 \times 2) \supseteq \{1, 3, 5, 6\}$ and $(2 \times 2) \times 7 = 4$. Suppose $2 \times 5 = 4$. Checking 5-table, we get $5 \times 5 = 3$. Then we can't define 5×7 since $N(5 \times 7) \supseteq \{1, 2, 3, 4, 5, 6\}$.

Example 5.40. G_{1083} is not a zero-divisor graph. G_{1083} can be defined by $N(1) = \{2, 5, 6, 7\}$, $N(2) = \{1, 3, 6\}$, $N(3) = \{2, 6, 7\}$, $N(4) = \{5, 6, 7\}$, $N(5) = \{1, 4, 7\}$, $N(6) = \{1, 2, 3, 4, 7\}$, $N(7) = \{1, 3, 4, 5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	1	1	0	0	0
2	0		0	7		0	7
3	1	0		1	6	0	0
4	1	7	1		0	0	0
5	0		6	0		6	0
6	0	0	0	0	6	0	0
7	0	7	0	0	0	0	0

We know $2 \times 5 \in \{6, 7\}$. Suppose $2 \times 5 = 6$. Checking 2-table, we can't define 2×2 since $N(2 \times 2) \supseteq \{1, 3, 5, 6\}$ and $(2 \times 2) \times 4 = 7$. Suppose $2 \times 5 = 7$. Checking 5-table, we can't define 5×5 since $N(5 \times 5) \supseteq \{1, 2, 4, 7\}$ and $(5 \times 5) \times 3 = 6$.

Example 5.41. G_{1120} is not a zero-divisor graph. G_{1120} can be defined by $N(1) = \{2, 3, 4, 5, 6\}$, $N(2) = \{1, 4, 5\}$, $N(3) = \{1, 4, 6\}$, $N(4) = \{1, 2, 3, 5, 6\}$, $N(5) = \{1, 2, 4, 6, 7\}$, $N(6) = \{1, 3, 4, 5, 7\}$, $N(7) = \{5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0	0	0	0	
2	0			0	0	6	
3	0			0	5	0	
4	0	0	0		0	0	
5	0	0	5	0		0	0
6	0	6	0	0	0		0
7					0	0	

We know $2 \times 3 \in \{1, 4, 5, 6\}$. Suppose $2 \times 3 = 1$. Checking the 2-table, we get $2 \times 2 = 6$. It follows that $N(2 \times 7) \supseteq \{1, 2, 4, 5, 6\}$. Hence $2 \times 7 \in \{1, 4, 5\}$. But $2 \times 7 \neq 1$ otherwise $7 \times 1 = 7 \times (2 \times 3) = (7 \times 2) \times 3 = 1 \times 3 = 0$. This is a contradiction. If $2 \times 7 = 4$, then $7 \times 1 = 7 \times (2 \times 3) = (7 \times 2) \times 3 = 4 \times 3 = 0$. Contradiction again. If $2 \times 7 = 5$, then $7 \times 1 = 7 \times (2 \times 3) = (7 \times 2) \times 3 = 5 \times 3 = 5$. But $1 \times 7 \neq 5$ since $N(1) \cup N(7) = N(1)$ and $N(1)$ is not a subset of $N(5)$.

Suppose $2 \times 3 = 4$. Checking 2-table, we get $2 \times 2 = 6$. It follows that $2 \times 7 = 5$. But then $7 \times 4 = 7 \times (2 \times 3) = (7 \times 2) \times 3 = 5 \times 3 = 5$. This is not true because $N(4) \cup N(7) = N(4)$ and $N(4)$ is not a subset of $N(5)$.

Suppose $2 \times 3 = 5$. Checking 3-table, we get $3 \times 3 = 3$. It follows that $3 \times 7 = 5$. Checking 7-table, one gets $N(7 \times 7) \supseteq \{3, 5, 6\}$. It follows that $7 \times 7 \in \{0, 1, 4, 6\}$. If $7 \times 7 = 0$ or $7 \times 7 = 1$, we get $4 \times 7 = 0$. This is a contradiction. If $7 \times 7 = 4$ or $7 \times 7 = 6$, we get $N(1 \times 7) \supseteq \{2, 3, 4, 5, 6, 7\}$. Hence we can't define 1×7 .

Suppose $2 \times 3 = 6$. Checking 2-table, we get $2 \times 2 = 2$. It follows that $2 \times 7 = 6$. Checking 7-table, we get $7 \times 7 \in \{0, 1, 4, 5\}$. If $7 \times 7 \in \{0, 4, 5\}$, then we can't define 1×7 since $N(1 \times 7) \supseteq \{2, 3, 4, 5, 6, 7\}$. If $7 \times 7 = 1$, then we can't define 4×7 since $N(4 \times 7) \supseteq \{1, 2, 3, 5, 6, 7\}$.

Example 5.42. The connected graph G_{1130} which is defined by $N(a) = \{b, d, e, f\}$, $N(b) = \{a, c, d, e, g\}$, $N(c) = \{b, d\}$, $N(d) = \{a, b, c, e, g\}$, $N(e) = \{a, b, d, f, g\}$, $N(f) = \{a, e, g\}$, $N(g) = \{b, d, e, f\}$ is not a zero-divisor graph. If we remove g , we get graph (4) in Theorem 5.2.

We have $ce = e$. It follows $c^2 \neq a$. Otherwise $c^2e = ae = 0 \neq e = c(ce)$. Similarly, $c^2 \neq g$. If $c^2 = c$, then $cf = e$. ($cf \neq b$. Otherwise $c^2f = cf = b \neq 0 = c(cf)$. $cf \neq d$. Otherwise $c^2f = cf = d \neq 0 = c(cf)$.) If $cf = e$, then $f^2 \neq f$. Otherwise, $cf^2 = cf = e \neq 0 = (cf)f$.

•	a	b	c	d	e	f	g
a		0		0	0	0	
b	0		0	0	0		0
c		0		0	e		
d	0	0	0		0		0
e	0	0	e	0		0	0
f	0				0		0
g		0		0	0	0	

c	ac	0	c^2	0	e	cf	cg
ac		0		0	0	0	
0	0	0	0	0	0	0	0
c^2		0		0	e		
0	0	0	0	0	0	0	0
e	0	0	e	0		0	0
cf	0	0		0	0		0
cg		0		0	0	0	

We get $c^2 = c$, or $c^2 = e$.

If $c^2 = e$, then

c	ac	0	e	0	e	cf	cg
ac		0	0	0	0	0	
0	0	0	0	0	0	0	0
e	0	0	e	0	e	0	0
0	0	0	0	0	0	0	0
e	0	0	e	0	e	0	0
cf	0	0	0	0	0		0
cg		0	0	0	0	0	

It follows that $N(ac) \supset \{b, c, d, e, f\}$. Hence we can not define ac . We get $c^2 = c$. It follows $cf = e$, $e^2 = 0$.

f	0	bf	e	df	0	f^2	0
0	0	0	0	0	0	0	0
bf	0		0	0	0		0
e	0	0	e	0	0	0	0
df	0	0	0		0		0
0	0	0	0	0	0	0	0
f^2	0		0		0		0
0	0	0	0	0	0	0	0

We know $f^2 \in \{0, b, d\}$. If $f^2 = 0$, then $N(bf) \supset \{a, c, d, e, f, g\}$, we can't define bf . If $f^2 = b$, then $N(df) \supset \{a, b, c, e, f, g\}$, can't define df . If $f^2 = d$, then $N(bf) \supset \{a, c, d, e, f, g\}$, we can't define bf .

Example 5.43. G_{1146} is not a zero-divisor graph. G_{1146} can be defined by $N(1) = \{2, 3, 4, 5\}$, $N(2) = \{1, 3, 7\}$, $N(3) = \{1, 2, 4, 6, 7\}$, $N(4) = \{1, 3, 5, 6, 7\}$, $N(5) = \{1, 4, 7\}$, $N(6) = \{3, 4, 7\}$, $N(7) = \{2, 3, 4, 5, 6\}$. We have the following table.

•	1	2	3	4	5	6	7
1		0	0	0	0		7
2	0		0	4			0
3	0	0		0	3	0	0
4	0	4	0		0	0	0
5	0		3	0			0
6			0	0			0
7	7	0	0	0	0	0	0

We know $2 \times 5 \in \{3, 4\}$. Suppose $2 \times 5 = 3$. Checking 2-table, we get $2 \times 6 = 4$ since $N(2 \times 6) \supseteq \{1, 3, 4, 5, 6, 7\}$. Since $(2 \times 2) \times 5 = 2 \times (2 \times 5) = 2 \times 3 = 0$ and $(2 \times 2) \times 6 = 2 \times (2 \times 6) = 2 \times 4 = 4$. It follows that $2 \times 2 \neq 2$ or $2 \times 2 \neq 4$. Then we can't define 2×2 .

Suppose $2 \times 5 = 4$. Checking 5-table, we get $5 \times 6 = 3$ since $N(5 \times 6) \supseteq \{1, 2, 3, 4, 7\}$. We also get $3 \times 3 = (5 \times 6) \times 3 = 5 \times (6 \times 3) = 5 \times 0 = 0$. Since $(5 \times 5) \times 3 = (5 \times 3) \times 5 = 3 \times 5 = 3$ and $(5 \times 5) \times 2 = (5 \times 2) \times 5 = 4 \times 5 = 0$, we have $5 \times 5 \neq 3$ or $5 \times 5 \neq 5$. Hence we can't define 5×5 .

Example 5.44. G_{1177} is not a zero-divisor graph. G_{1177} can be defined by $N(1) = \{2, 3, 6\}$, $N(2) = \{1, 3, 4, 5, 6\}$, $N(3) = \{1, 2, 4, 5, 6\}$, $N(4) = \{2, 3, 5, 6, 7\}$, $N(5) = \{2, 3, 4, 6, 7\}$, $N(6) = \{1, 2, 3, 4, 5\}$, $N(7) = \{4, 5\}$. We know $1 \times 4 \in \{4, 5\}$, $1 \times 5 \in \{4, 5\}$, $2 \times 7 \in \{2, 3, 6\}$, $2 \times 3 \in \{2, 3, 6\}$, and $6 \times 7 \in \{2, 3, 6\}$.

Checking 1-table, we get $1 \times 7 \in \{4, 5\}$ otherwise we can't define 1×1 .

Checking 7-table, we can't define 7×7 .

6. NON ZERO-DIVISOR GRAPHS WHICH ARE EITHER DIS-CONNECTED OR CONNECTED BUT NOT SATISFYING THE \star CONDITION

6.1. Dis-connected graphs are not zero-divisor graphs.

$G_{209} - G_{269}, G_{275}, G_{277}, G_{281} - G_{283}, G_{285}, G_{287} - G_{313}, G_{323}, G_{330},$
 $G_{335}, G_{345} - G_{347}, G_{352}, G_{354} - G_{378}, G_{387}, G_{397}, G_{407}, G_{417} - G_{418}, G_{451} - G_{472}, G_{496},$
 $G_{502}, G_{582} - G_{597}, G_{611}, G_{731} - G_{739}, G_{745}, G_{879} - G_{883}, G_{1010} - G_{1011}, G_{1107},$
 $G_{1172}.$

6.2. connected but not satisfying the \star condition.

$G_{273} - G_{274}, G_{276}, G_{278} - G_{280}, G_{284}, G_{286}, G_{318}, G_{320} - G_{321}, G_{324} - G_{329}, G_{331} -$
 $G_{334}, G_{336} - G_{344}, G_{348} - G_{351}, G_{353}, G_{385} - G_{386}, G_{389}, G_{391},$
 $G_{394} - G_{396}, G_{398} - G_{404}, G_{406}, G_{408} - G_{410}, G_{412} - G_{416}, G_{419} - G_{450}, G_{484}, G_{487} -$
 $G_{489}, G_{491} - G_{492}, G_{494} - G_{495}, G_{497} - G_{501}, G_{505} - G_{506}, G_{508} - G_{512}, G_{514} -$
 $G_{521}, G_{523} - G_{524}, G_{526} - G_{550}, G_{552} - G_{581}, G_{605}, G_{608} - G_{610}, G_{615},$
 $G_{621} - G_{623}, G_{625}, G_{628}, G_{630}, G_{632}, G_{634}, G_{636} - G_{638}, G_{640} - G_{666},$
 $G_{673} - G_{676}, G_{679} - G_{730}, G_{744}, G_{756}, G_{760} - G_{761}, G_{763}, G_{765}, G_{768} - G_{771}, G_{773} -$
 $G_{774}, G_{776} - G_{779}, G_{781} - G_{785}, G_{787} - G_{789}, G_{797}, G_{802},$
 $G_{804}, G_{806} - G_{807}, G_{809} - G_{811}, G_{816} - G_{831}, G_{833} - G_{871}, G_{873} - G_{878}, G_{892}, G_{895},$
 $G_{900} - G_{901}, G_{903} - G_{905}, G_{910} - G_{912}, G_{926}, G_{931}, G_{935} - G_{937}, G_{940} -$
 $G_{943}, G_{945} - G_{947}, G_{949}, G_{954} - G_{956}, G_{958} - G_{971}, G_{973} - G_{974}, G_{976} - G_{1006},$
 $G_{1021} - G_{1023}, G_{1033}, G_{1038} - G_{1042}, G_{1051}, G_{1054} - G_{1055}, G_{1058}, G_{1061}, G_{1063} -$
 $G_{1066}, G_{1068} - G_{1071}, G_{1073} - G_{1076}, G_{1082}, G_{1084}, G_{1086} - G_{1087},$
 $G_{1089} - G_{1105}, G_{1112}, G_{1127}, G_{1133}, G_{1136}, G_{1153} - G_{1156}, G_{1158} - G_{1162}, G_{1164} -$
 $G_{1168}, G_{1170} - G_{1171}, G_{1201}, G_{1204}, G_{1209}, G_{1211} - G_{1212},$

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